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Automorphism group of the moduli space of parabolic vector bundles over a curve

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Dedicated to my mother Adela

Dedicada a mi madre Adela

Abstract

The main objective of this thesis is the computation of the automorphism group of the moduli space of parabolic vector bundles over a smooth complex projective curve.

We will start by defining the notion of a parabolic Λ -module – a module over a sheaf of rings of differential operators Λ with a parabolic structure at certain marked points – and building their moduli space. This will provide us a common theoretical framework that allows us to work with several kinds of moduli spaces of bundles with parabolic structure such parabolic vector bundles, parabolic (L -twisted) Higgs bundles, parabolic connections or parabolic λ -connections. As an application, we build the parabolic Hodge moduli space and the parabolic Deligne–Hitchin moduli space.

Then, we will address the computation of the automorphism group of the moduli space of parabolic bundles. Let X and X' be irreducible smooth complex projective curves with sets of marked points $D \subset X$ and $D' \subset X'$ and genus $g \geq 6$ and $g' \geq 6$ respectively. Let $\mathcal{M}(X, r, \alpha, \xi)$ be the moduli space of rank r stable parabolic vector bundles on (X, D) with parabolic weights α and determinant ξ . We classify the possible isomorphisms $\Phi : \mathcal{M}(X, r, \alpha, \xi) \xrightarrow{\sim} \mathcal{M}(X', r', \alpha', \xi')$. First, a Torelli type theorem is proved, implying that for Φ to exist it is necessary that $(X, D) \cong (X', D')$ and $r = r'$. Then we prove that the possible isomorphisms are generated by automorphisms of the pointed curve (X, D) , tensorization with suitable line bundles, dualization of parabolic vector bundles and Hecke transformations at the parabolic points. These results are extended to birational equivalences $\Phi : \mathcal{M}(X, r, \alpha, \xi) \dashrightarrow \mathcal{M}(X', r', \alpha', \xi')$ which are defined over “big” open subsets. The particular case of “concentrated” weights (corresponding to “small” stability parameters) is studied further. In this case Hecke transformations give rise to birational morphisms that do not extend to automorphisms of the moduli space. Moreover, an analysis of the stability chambers for the weights α allows us to determine an explicit computable presentation of the group of automorphisms of the moduli space for arbitrary generic weights.

Finally, the automorphism group of the moduli space of framed bundles over a smooth complex projective curve X of genus $g > 2$ with a framing over a point $x \in X$ is also described. It is shown that this group is generated by pullbacks using automorphisms of the curve X that fix the marked point x , tensorization with certain line bundles over X and the action of $\mathrm{PGL}_r(\mathbb{C})$ by composition with the framing.

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Key words and phrases. Parabolic vector bundle, moduli space, automorphism group, Extended Torelli theorem, birational geometry, stability chambers, parabolic Λ -module, parabolic connection, parabolic Hodge moduli space, Framed bundles.

Resumen

El principal objetivo de esta tesis es el cálculo del grupo de automorfismos del espacio de moduli de fibrados parabólicos sobre una curva compleja proyectiva suave.

Comenzaremos definiendo la noción de Λ -módulo parabólico – un módulo sobre un haz de anillos de operadores diferenciales Λ con una estructura parabólica en ciertos puntos prefijados – y construyendo su correspondiente espacio de moduli. Esto nos proporcionará un marco teórico común para trabajar con diferentes tipos de espacios de moduli de fibrados con estructuras parabólicas tales como fibrados vectoriales parabólicos, fibrados de Higgs (L -twistados) parabólicos, conexiones parabólicas o λ -conexiones parabólicas. Como aplicación, construimos el espacio de moduli de Hodge parabólico y el espacio de Deligne–Hitchin parabólico.

A continuación, afrontaremos el cálculo del grupo de automorfismos del espacio de moduli de fibrados parabólicos. Sean X y X' curvas complejas suaves irreducibles de género $g \geq 6$ y $g' \geq 6$ con un conjunto de puntos marcados $D \subset X$ y $D' \subset X'$ respectivamente. Sea $\mathcal{M}(X, r, \alpha, \xi)$ el espacio de moduli de fibrados parabólicos estables de rango r , pesos parabólicos α y determinante ξ sobre (X, D) . Buscamos clasificar los posibles isomorfismos $\Phi : \mathcal{M}(X, r, \alpha, \xi) \xrightarrow{\sim} \mathcal{M}(X', r', \alpha', \xi_0)$. En primer lugar, probamos un teorema tipo Torelli que implica que para que Φ exista es necesario que $(X, D) \cong (X', D')$ y $r = r'$. Entonces probamos que los posibles isomorfismos están generados por automorfismos de la curva marcada (X, D) , tensorización por fibrados de línea adecuados, dualización de fibrados parabólicos y transformaciones de Hecke en los puntos parabólicos. Estos resultados se extienden a equivalencias birracionales $\Phi : \mathcal{M}(X, r, \alpha, \xi) \dashrightarrow \mathcal{M}(X', r', \alpha', \xi')$ que están definidas sobre subconjuntos abiertos “grandes”. El caso particular de pesos “concentrados” (correspondientes a parámetros de estabilidad “pequeños”) es estudiado en mayor profundidad. En este caso, las transformaciones de Hecke dan lugar a aplicaciones biracionales que no se extienden a automorfismos del espacio de moduli. Por otro lado, mediante el análisis de las cámaras de estabilidad para los pesos α podemos determinar de forma explícita y computable una presentación para el grupo de automorfismo del espacio de moduli para pesos genéricos arbitrarios.

Finalmente, describimos el grupo de automorfismos del moduli de fibrados marcados sobre una curva suave proyectiva compleja X de género $g > 2$, con un marcado sobre un punto $x \in X$. Se demuestra que este grupo está generado por pullbacks con respecto a automorfismos de la curva X que fijan el punto marcado x , tensorización con ciertos fibrados de línea sobre X y la acción de $\mathrm{PGL}_r(\mathbb{C})$ por composición con el marcado.

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Palabras y conceptos clave. Fibrado parabólico, espacio de móduli, grupo de automorfismos, teorema de Torelli extendido, geometría birracional, cámaras de estabilidad, Λ -módulo parabólico, conexión parabólica, espacio de moduli de Hodge parabólico, fibrado marcado.

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Chapter 1

Introduction

The results contained in this thesis can be divided in two related blocks. The first part of the thesis (mainly contained in Chapter 3) is devoted to the development of a common framework for studying and constructing moduli spaces of parabolic vector bundles with additional structures, namely parabolic vector bundle, parabolic Higgs bundles, parabolic connections, etc. This is achieved by defining a parabolic analogue of the notion of Λ -module introduced by Simpson [Sim95]. Roughly speaking, a parabolic Λ -module is a parabolic vector bundle E with an action of some sheaf of rings of differential operators Λ which respects the filtration given by the parabolic structure. More precisely, if X is a smooth complex projective curve with a finite set of marked points $D \subset X$ and Λ is a sheaf of rings of differential operators on X , then a parabolic Λ -module is a vector bundle E over X together with

- A filtration by linear subspaces

$$E|_x = E_{x,1} \supsetneq E_{x,2} \supsetneq \cdots \supsetneq E_{x,l_x} \supsetneq E_{x,l_x+1} = 0$$

together with a system of real weights $0 \leq \alpha_{x,1} < \cdots < \alpha_{x,l_x} < 1$ for each $x \in D$

- An action $\Lambda \otimes E \rightarrow E$ preserving the filtration in the following way. For each $x \in D$ let

$$E = E_x^1 \supsetneq E_x^2 \supsetneq \cdots \supsetneq E_x^{l_x} \supsetneq E_x^{l_x+1} = E(-x)$$

be the induced filtration of E by subsheaves given by

$$0 \longrightarrow E_x^i \longrightarrow E \longrightarrow E|_x/E_{x,i} \longrightarrow 0$$

Then the image of $\Lambda \otimes E_x^i$ under the morphism $\Lambda \otimes E \longrightarrow E$ lies in E_x^i for every $i = 1, \dots, l_x + 1$.

In particular, if Λ is simply the sheaf of rings \mathcal{O}_X , then a parabolic \mathcal{O}_X -module is just a parabolic vector bundle. If L is any line bundle over X , a parabolic $\mathrm{Sym}^\bullet(L)$ -bundle is an L -twisted parabolic Higgs bundle and if we take Λ as the sheaf $\Lambda^{\mathrm{DR}, \log D}$ of logarithmic operators with simple poles on D then parabolic $\Lambda^{\mathrm{DR}, \log D}$ -modules are parabolic connections (filtered logarithmic connections on (X, D)).

The main motivation for this study is to construct the moduli space of parabolic λ -connections, also called parabolic Hodge moduli space, and a parabolic analogue

of the Deligne–Hitchin moduli space. These moduli spaces were studied in detail through my masters thesis, culminating in the proof of a Torelli type theorem (see [AG18b] for details). Nevertheless, the existing constructions in the literature for similar moduli spaces did not cover the exact moduli problem treated in this case. On the other hand, the abstraction of Λ -modules allows us to construct the desired moduli through an appropriate choice of the sheaf of rings of differential operators Λ .

We define a natural notion of stability for parabolic Λ -modules which specializes to the well known stability conditions for the previous examples, and we construct the moduli space of parabolic Λ -modules. Moreover, we define a notion of residue of a parabolic Λ -module, mirroring the residue of a logarithmic connection at a parabolic point. An appropriate control of the residue of a logarithmic connection is very important, as it serves a crucial role in the Simpson correspondence between parabolic connections, parabolic Higgs bundles and filtered local systems [Sim90]. In particular, imposing restrictions on the admissible residues of a parabolic λ -connection is mandatory for our construction of the parabolic Deligne–Hitchin moduli space.

Moreover, the inclusion of this chapter in the thesis serves to improve the self-containness of the work, as the provided construction for moduli spaces of parabolic Λ -modules gives an explicit alternative common construction for the parabolic moduli spaces appearing in the rest of the thesis. Although moduli spaces of parabolic vector bundles or parabolic Higgs bundles were already built with other methods in the literature [MS80, MY92, Yok93], the reader may refer to this section for additional details on the structure of the moduli spaces. For instance, the existence of the universal family over the moduli space of full flag parabolic vector bundles is needed in a latter chapter. This result was originally proven by Boden and Yokogawa in [BY99], but the reader might find another proof within this part of the thesis, as we show a similar more general lemma stating the existence of a universal bundle over the moduli space of parabolic Λ -modules for any choice of Λ under certain conditions on the flag type.

In the second part of the thesis, we engage the main objective of this work: the study of the automorphism groups of moduli spaces. In particular, we focus on the classification of the symmetries of the moduli space of parabolic vector bundles. The automorphisms of the moduli space of vector bundles in the non-parabolic case were originally classified by Kouvidakis and Pantev [KP95]. They proved that the following two transformations generate the automorphism group of the moduli space $\mathcal{M}(r, \xi)$ of stable vector bundles over X with rank r and determinant ξ over X . Given an automorphism $\sigma : X \rightarrow X$

1. Send $E \rightarrow X$ to $L \otimes \sigma^* E$, where L is a line bundle over X with $L^r \otimes \sigma^* \xi \cong \xi$
2. Send E to $L \otimes \sigma^*(E^\vee)$, where L is a line bundle satisfying $L^r \otimes \sigma^* \xi^{-1} \cong \xi$

This result was also proved by Hwang and Ramanan [HR04] through the study of Hecke curves on the moduli space. Later on, Biswas, Gómez and Muñoz simplified their proof [BGM13], which allowed them to extend it to the moduli space of symplectic bundles [BGM12]. One of the main ideas behind the simplified proof in [BGM13] is to be able to recover the Hitchin discriminant from the isomorphism class of $\mathcal{M}(r, \xi)$ as the union of the complete rational curves in $T^*\mathcal{M}(r, \xi)$. This,

combined with an argument involving the standard \mathbb{C}^* -action on the Hitchin space induced by dilations on $T^*\mathcal{M}(r, \xi)$, allows us to determine the isomorphism class of X , thus proving a Torelli type theorem. On the other hand, they proved that if an automorphism $\Phi : \mathcal{M}(r, \xi) \rightarrow \mathcal{M}(r, \xi)$ sends E to $\Phi(E) = E'$, then for generic E , then there exists an isomorphism of Lie algebra bundles

$$\mathrm{End}_0(E) \cong \mathrm{End}_0(E')$$

Moreover, such an automorphism is only possible if E' is obtained from E by one of the previously mentioned transformations, leading to the result.

In the parabolic scenario the presence of the flags at the parabolic points changes significantly the geometry of the moduli space and the structure of its automorphism group. In particular, the additional data of a flag at the parabolic points allows us to define more transformations on the moduli space of parabolic vector bundles that do not come from an automorphism of the moduli space of vector bundles. More precisely, for each parabolic vector bundle (E, E_\bullet) we can use the steps of the filtration

$$E|_x = E_{x,1} \supsetneq E_{x,2} \supsetneq \cdots \supsetneq E_{x,r} \supsetneq 0$$

to perform a Hecke transformation on E at x with respect to one of the steps $E_{x,i}$

$$0 \longrightarrow E_x^i \longrightarrow E \longrightarrow E|_x/E_{x,i} \longrightarrow 0$$

and endow E_x^i a filtration “rotating” the parabolic structure of E at x . We denote the corresponding quasiparabolic bundle as $\mathcal{H}_x^i(E, E_\bullet)$. The analysis on Hecke transformations was pioneered in [NR778] and [HR04] and this “rotation” procedure has been used in the literature to perform correspondences between moduli spaces of parabolic bundles [BY99, IIS06b, Ina13]. Then, we prove that the automorphisms of the moduli space are obtained as a combination of the following four types of transformations

- Taking pullback with respect to an automorphism $\sigma : X \rightarrow X$ that fixes the set of parabolic points D (but not necessarily fixes every point in D) $(E, E_\bullet) \mapsto \sigma^*(E, E_\bullet)$
- Tensoring with a line bundle $(E, E_\bullet) \mapsto (E, E_\bullet) \otimes L$
- Dualization $(E, E_\bullet) \mapsto (E, E_\bullet)^\vee$
- Hecke transformations $(E, E_\bullet) \mapsto \mathcal{H}_x(E, E_\bullet)$ with respect to the subspace $E_{x,2} \subset E|_x$ for some $x \in D$

The proof of this result will be based on generalizing to the parabolic scenario some of the key ideas previously described from [BGM12] and [BGM13], although a deeper analysis on the geometry of the moduli space, and specially on the geometry of the Hitchin discriminant (the set of points in the Hitchin space whose corresponding spectral curves are singular) will be needed. Moreover, contrary to the non-parabolic case, in the parabolic scenario there exists a stability parameter α for the moduli space. Let $\mathcal{M}(r, \alpha, \xi)$ be the moduli space of parabolic vector bundles with rank r , determinant ξ and system of weights α . As α varies we obtain different moduli spaces

of parabolic vector bundles and, while the previously described transformations can be described in families of quasi-parabolic vector bundles, it is not trivial to see whether they preserve the stability of the bundles. Instead, what we observe is that for every system of weights α and every basic transformation T obtained as a combinations of the previous ones (pullback, tensorization, dualization and Hecke), there exists another possibly distinct system of weights $T(\alpha)$ such that a parabolic vector bundle (E, E_\bullet) is α -stable if and only if its image by the transformation $T(E, E_\bullet)$ is $T(\alpha)$ -stable. The problem of deciding if T induces an automorphism of $\mathcal{M}(r, \alpha, \xi)$ or not then relies on determining if all $T(\alpha)$ -stable parabolic vector bundles are also α -stable or whether, on the contrary, there is some $T(\alpha)$ -stable (E, E_\bullet) which is not α -stable. The solution of this latter problem for generic α involves a deep analysis on the stability chamber structure for the moduli space.

This will motivate a change in the point of view of our analysis on the automorphism group contrasting with the approaches in [KP95], [HR04] or [BGM12, BGM13]. Instead of analyzing a single moduli space $\mathcal{M}(X, r, \alpha, \xi)$ for fixed parameters and study its automorphisms, we will organically work with an isomorphism $\Phi : \mathcal{M}(X, r, \alpha, \xi) \rightarrow \mathcal{M}(X', r', \alpha', \xi')$ between moduli spaces with possibly different parameters and then work to determine which combinations of parameters can really correspond to isomorphic moduli spaces.

First of all, we identify the image of the Hitchin discriminant as the image of the complete rational curves projected to the Hitchin space by the Hitchin map

$$T^*\mathcal{M}(r, \alpha, \xi) \longrightarrow W = \bigoplus_{j=2}^r H^0(X, K^j D^{j-1})$$

Analyzing the structure of the image of the Hitchin discriminant and using an argument involving the \mathbb{C}^* -action on the Hitchin space W , we prove a Torelli type theorem for the moduli space of parabolic vector bundles. This Torelli type theorem is of particular interests, because it generalizes deeply the previously known Torelli theorem for the moduli space of parabolic vector bundles proven by Balaji, del Baño and Biswas [BdBnB01]. The result in [BdBnB01] is only valid for rank 2, degree 1 and small systems of weights, while our new result is valid for arbitrary rank, degree and generic systems of weights. In achieving so, it unlocks several Torelli results that were known only for rank 2 and small parabolic weights due to their dependence on the usage of the Torelli theorem in [BdBnB01]. For instance, the newly obtained Torelli theorem, combined with our previous results on Torelli theorems for the parabolic Higgs moduli space, the parabolic Hodge moduli space and the parabolic Deligne–Hitchin moduli space, lead to the generalization of the latter results directly to arbitrary rank, degree and generic weights.

Then we will deepen in the analysis of the Hitchin map W and the Hitchin discriminant \mathcal{D} . We found that instead of studying the space of singular spectral curves \mathcal{D} alone, there is another set $\mathcal{N} \subset \mathcal{D}$ whose geometry gives us a more suitable way to obtain information about W than \mathcal{D} itself. This space \mathcal{N} is the set of non-reduced spectral curves, i.e., the set of points in W whose corresponding spectral curve has at least a non-reduced component. To give a glimpse of how rich the geometry of \mathcal{N} is, we prove that if we have a \mathbb{C}^* -equivariant automorphism $f : W \rightarrow W$ such that $f(\mathcal{N}) = \mathcal{N}$, then f must be linear and must decompose diagonally as

$f = (f_2, \dots, f_r)$ with $f_j : H^0(K^j D^{j-1}) \rightarrow H^0(K^j D^{j-1})$. Moreover, we prove that we can reconstruct $\mathcal{N} \subset W$ geometrically from the geometry of \mathcal{D} , so it can be intrinsically characterized inside the Hitchin space.

Using this, we prove that if $\Phi : \mathcal{M}(r, \alpha, \xi) \rightarrow \mathcal{M}(r, \alpha', \xi')$ is an isomorphism sending a generic (E, E_\bullet) to a generic (E', E'_\bullet) , then there is a Lie algebra bundle isomorphism

$$\mathrm{PEnd}_0(E, E_\bullet) \cong \mathrm{PEnd}_0(E', E'_\bullet)$$

and we show that the latter implies that (E', E'_\bullet) can be indeed obtained from (E, E_\bullet) through the combination of the previously described transformations.

At this point is when we really encounter the problems originated from the variations of the stability parameters. Even if for a generic point (E, E_\bullet) its image (E', E'_\bullet) is given by a basic transformation $(E', E'_\bullet) = T(E, E_\bullet)$ it is still left to prove that we can extend these basic transformations to the whole moduli space. In contrast to the non-parabolic case, basic transformations do not always induce well defined self-maps on the whole space, as we know that the stability $T(\alpha)$ of the resulting bundles may not belong to the same stability chamber as α . Instead, they can only be extended to the locus of parabolic bundles which are both α -stable and $T^{-1}(\alpha)$ -stable.

We prove that this locus is a big open subset of $\mathcal{M}(r, \alpha, \xi)$, in the sense that its complement has codimension at least 3. Therefore, in general, basic transformations only induce birational self-equivalences $\mathcal{M}(X, r, \alpha, \xi) \dashrightarrow \mathcal{M}(X, r, \alpha, \xi)$, instead of automorphisms of the moduli space $\mathcal{M}(X, r, \alpha, \xi)$. Nevertheless, we observe that they induce a specially regular type of birational maps, as we know that they define isomorphisms between “big” dense subsets, i.e., between subsets whose complements have codimension at least k for some $k > 1$. In general, we will call this type of birational maps k -birational equivalences. In general, the study of the k -birational equivalence class of a variety is important, as it preserves certain types of geometrical invariants of the variety that, in general, are not preserved by mere birational equivalences. For instance, the Picard group is preserved under 2-birational equivalences, but not under (1-)birational maps.

Then we jump from the classification of the isomorphisms between moduli spaces of parabolic vector bundles to the classification of k -birational equivalences between the moduli spaces. We prove that, in fact, the 3-birational equivalence class of the moduli space $\mathcal{M}(X, r, \alpha, \xi)$ is enough to determine the isomorphism class of the marked curve (X, D) , thus obtaining a k -birational version of the Torelli theorem for the moduli space of parabolic vector bundles. Moreover, this k -birational version actually has a simple reciprocal, as we prove that two moduli spaces $\mathcal{M}(X, r, \alpha, \xi)$ and $\mathcal{M}(X', r', \alpha', \xi')$ are isomorphic if and only if $r = r'$ and $(X, D) \cong (X', D')$ and we prove that the isomorphisms between moduli spaces are defined precisely by basic transformations.

On the other hand, we engage the actual classification of the parabolic chambers and the structure of the wall crossings of the moduli space of parabolic vector bundles. Using Brill-Noether theory and the characterization of the stratification of the moduli space of parabolic vector bundles in terms of the Segre invariant developed by Biswas and Bhosle [BB05], we give an explicit computable characterization of the stability chambers for high genus curves, as well as a qualitative description of

the evolution of the k -birational automorphism and the automorphism groups as we change k and increase the genus of the curve. The analysis of the concentrated chamber (i.e., the stability chamber corresponding to “small” parameters α such that α -stability of a parabolic vector bundle (E, E_\bullet) is essentially equivalent to the stability of the underlying vector bundle E) is further developed.

Finally, using the experience and strategies learned through the study of the isomorphisms between moduli spaces of parabolic vector bundles, we will analyze the automorphism group of the moduli space of framed bundles. Let X be a compact connected Riemann surface with a marked point $x \in X$. A framed bundle is a vector bundle E over X together with a “framing” over the point $x \in X$, i.e., a nonzero linear map $\alpha : E|_x \rightarrow \mathbb{C}^r$. Observe that, in general we do not ask the framing to be an isomorphism, we just require it to be nonzero. Framed bundles were defined by Donaldson as a tool to study the moduli space of instantons on \mathbb{R}^4 [Don84]. Morally, they can be seen as a “universal” or rigidified parabolic bundle in the following sense. Let us fix once and for all a filtration on \mathbb{C}^r . Then taking the pullback by the framing $\alpha : E|_x \rightarrow \mathbb{C}^r$ gives us canonically a filtration on $E|_x$ for every framed bundle (E, α) . All possible nontrivial filtrations on $E|_x$ can be obtained this way when we range over the possible framings $\alpha : E|_x \rightarrow \mathbb{C}^r$. Therefore, it is natural that we could be able to transfer some of the ideas developed for parabolic bundles to this new framework.

The moduli space admits a natural $\mathrm{PGL}_r(\mathbb{C})$ -action, induced by “rotations” of the framing. For each $[G] \in \mathrm{PGL}_r(\mathbb{C})$, send each framed bundle (E, α) to

$$[G] \cdot (E, \alpha) = (E, G \circ \alpha)$$

Moreover, if $\sigma : X \rightarrow X$ is an automorphism of the curve X fixing the point $x \in X$, then for each framed bundle (E, α) , (σ^*E, α) is another bundle with a framing at x . Similarly, if L is a line bundle on X and $\alpha_L : L|_x \xrightarrow{\sim} \mathbb{C}$ is a trivialization at x , then for each framed bundle (E, α) ,

$$(E \otimes L, \alpha \cdot \alpha_L)$$

is a framed bundle on (X, x) . It is easy to check that these transformations always preserve the stability of the framed bundle, so suitable combinations of them induce automorphisms of the moduli space of framed bundles. In fact, we prove that the automorphism group of the moduli space of framed bundles is generated by suitable combinations of these three types of transformations.

1.1 Document structure

The rest of the document is structured in the following way. Chapter 2 corresponds to a state of the art analysis about the existence of moduli spaces of vector bundles enhanced with different structures (parabolic structures, Higgs fields, connections, framings, etc.), Torelli type theorems and the automorphism groups of such schemes. The main results presented in this work are then introduced in Chapters 3, 4 and 5. Chapter 3 is devoted to defining the notion of parabolic Λ -module and proving the existence of the corresponding moduli space, giving applications to the existence of the moduli space of parabolic λ -connections and the Deligne–Hitchin moduli space.

The analysis of the automorphisms of the moduli space of parabolic vector bundles and the moduli space of framed bundles are issued in Chapters 4 and 5 respectively. Chapter 4 also contains the classification of k -birational maps between moduli spaces of parabolic vector bundles and an analysis on the stability chambers of the moduli space. Particularly

- **Chapter 2** summarizes some of the most relevant results on the existence of moduli spaces of vector bundles enhanced with different structures: parabolic structures, Higgs fields, connections, framings, etc. Torelli type theorems for these spaces are also described and we review some known results concerning the structure of the symmetries of some of these schemes.
- In **Chapter 3** a parabolic analogue of Simpson's notion of Λ -module is defined. A natural stability condition for parabolic Λ -modules generalizing the usual one for parabolic Higgs bundles or parabolic connections is defined and the existence of a coarse moduli space of stable parabolic Λ -modules is proven. Moreover, we prove the existence of a universal family for the moduli space under some mild conditions. Finally, residual structures on parabolic Λ -modules are described and the moduli space of residual parabolic Λ -modules is built. We use this framework to provide an algebraic construction to the moduli space of parabolic connections and to construct a parabolic analogue for the Deligne–Hitchin moduli space over a curve.
- In **Chapter 4** A general Torelli theorem for the moduli space is proven in Section 4.3. Isomorphisms between moduli spaces of parabolic vector bundles are classified in Section 4.6 and the automorphism group of the moduli space is computed. We address an analogous Torelli theorem for k -birational maps and the classification of k -birational maps between moduli spaces of parabolic vector bundles in Section 4.7. Moreover, we perform an analysis on the stability chambers for the moduli space of parabolic vector bundles on high genus curves, in Section 4.9 that allows us to refine a explicit description of the automorphisms group. Finally, the case of concentrated weights (i.e., small stability parameters, for which the stability of the parabolic vector bundle is equivalent to the stability of the underlying vector bundle) is further analyzed in Section 4.8.
- The classification of isomorphisms between moduli spaces of framed bundles is addressed in **chapter 5**.
- **Chapters 6 and 7** expose the conclusions of our work (in English and Spanish respectively), remark the obtained results and contributions and describe future work research lines.
- **Appendix A** includes the proof of some basic lemmata concerning sheaves of bi-modules over possible non-commutative sheaves of rings.
- In **Appendix B** descriptions of the category of parabolic vector bundles from different formalisms are reviewed. A comparison between the different frameworks of parabolic bundles is performed and some useful results about parabolic vector bundles are provided.

1.2 Main results

In order to facilitate the navigation through the results presented in this thesis and to put them in the corresponding context, this section encloses a brief summary of statement of the main new theorems proved in this work, including references to their corresponding proofs.

- Development of the framework of parabolic Λ -modules
 - Existence of a moduli space of parabolic Λ -modules. Theorem 3.4.8 (see Section 3.1 for the definition of parabolic Λ -module)
 - Definition of residual construction for Λ -modules. Section 3.5.
 - Construction of the moduli space of residual parabolic Λ -modules. Theorem 3.5.3
 - Existence of a universal bundle on the moduli spaces of parabolic Λ -modules. Theorem 3.6.3 and Corollary 3.6.4
 - Construction of the parabolic Hodge moduli space 3.7.4
 - The Riemann–Hilbert map for parabolic connections is a biholomorphism. Theorem 3.8.7 (an alternative proof of this result can found in [Ina13])
- Classification of isomorphisms between moduli spaces of parabolic vector bundles
 - Torelli theorem for the moduli space of parabolic vector bundles for arbitrary rank, arbitrary fixed determinant and generic full flag weights. If $\mathcal{M}(X, r, \alpha, \xi) \cong \mathcal{M}(X', r', \alpha', \xi')$ then $r = r'$ and $(X, D) \cong (X', D')$. Theorem 4.3.6
 - Classification of the isomorphisms $\Phi : \mathcal{M}(X, r, \alpha, \xi) \rightarrow \mathcal{M}(X', r', \alpha', \xi')$ between moduli spaces of parabolic vector bundles. We prove that they are given by suitable basic transformations. Theorem 4.6.22 (see Section 4.4 for the definition of basic transformations)
 - Explicit computable description of the isomorphisms $\Phi : \mathcal{M}(X, r, \alpha, \xi) \rightarrow \mathcal{M}(X', r', \alpha', \xi')$ between moduli spaces of parabolic vector bundles for high genus curves. Theorem 4.9.8
 - “Refined” Torelli theorem. $\mathcal{M}(X, r, \alpha, \xi) \cong \mathcal{M}(X', r', \alpha', \xi')$ if and only if $r = r'$, there exists $\sigma : (X, D) \cong (X', D')$ and $(\deg(\xi), \alpha) \sim_{\Theta} (\deg(\xi'), \sigma^* \alpha')$ for some explicit equivalence relation \sim_{Θ} . Description of the space $\tilde{\Theta}/\mathcal{T}$ classifying isomorphism classes of moduli spaces of parabolic vector bundles on a marked curve (X, D) . Theorem 4.11.1
 - “Refined” Torelli theorem for the moduli space of parabolic vector bundles for arbitrary rank, arbitrary fixed degree and generic full flag weights. $\mathcal{M}(X, r, \alpha, d) \cong \mathcal{M}(X', r', \alpha', d')$ if and only if $r = r'$, there exists $\sigma : (X, D) \cong (X', D')$ and $(d, \alpha) \sim_{\Theta} (d', \sigma^* \alpha')$. Theorem 4.11.4.
 - 3-birational Torelli theorem. If $\Phi : \mathcal{M}(X, r, \alpha, \xi) \dashrightarrow \mathcal{M}(X', r', \alpha', \xi')$ is a 3-birational equivalence, then $r = r'$ and $(X, D) \cong (X', D')$. Theorem 4.7.5 (k -birational equivalences are defined in Definition 4.7.1)

- Reciprocal of the k -birational Torelli. For any α, α', ξ and $\xi', \mathcal{M}(X, r, \alpha, \xi) \stackrel{k\text{-bir}}{\cong} \mathcal{M}(X', r', \alpha', \xi')$ for some k depending on the genus (in particular, 3-bir for $g \geq 3$). Proposition 4.7.6
- 3-birational Torelli theorem for the moduli space of parabolic vector bundles with fixed degree. $\mathcal{M}(X, r, \alpha, d) \stackrel{3\text{-bir}}{\cong} \mathcal{M}(X', r', \alpha', d')$ if and only if $r = r'$ and $(X, D) \cong (X', D')$. Theorem 4.11.6.
- Refined Torelli theorem for the moduli space of vector bundles. $\mathcal{M}(X, r, \xi) \cong \mathcal{M}(X', r', \xi')$ if and only if $r = r'$, $X \cong X'$ and $\deg(\xi) \equiv \pm \deg(\xi') \pmod{r}$. Theorem 4.11.2 (obtained working on the proof of [BGM13])
- Computation of automorphism groups for the moduli spaces of parabolic vector bundles

- The group of basic transformations \mathcal{T} is isomorphic to

$$\mathcal{T} \cong \left((\mathbb{Z}^{|D|} \times \text{Pic}(X)) / \mathcal{G}_D \right) \rtimes (\text{Aut}(X, D) \times \mathbb{Z}/2\mathbb{Z})$$

where $\mathcal{G}_D < \mathbb{Z}^{|D|} \times \text{Pic}(X)$ is a (normal) subgroup isomorphic to $(r\mathbb{Z})^{|D|}$. Proposition 4.4.9.

- The group of automorphisms of the moduli space of parabolic vector bundles is

$$\text{Aut}(\mathcal{M}(X, r, \alpha, \xi)) = \left\{ T \in \mathcal{T} \left| \begin{array}{l} T(\xi) \cong \xi \\ T(\alpha) \text{ is in the same stability chamber as } \alpha \end{array} \right. \right\} < \mathcal{T}$$

Theorem 4.6.24

- Computable refinement of the previous result for high genus

$$\text{Aut}(\mathcal{M}(X, r, \alpha, \xi)) = \left\{ T \in \mathcal{T} \left| \begin{array}{l} T(\xi) \cong \xi \\ \overline{M}(r, T(\alpha), \deg(\xi)) = \overline{M}(r, \alpha, \deg(\xi)) \end{array} \right. \right\} < \mathcal{T}$$

Theorem 4.6.24

- Explicit description of the automorphisms for concentrated weights. If α is concentrated

$$\text{Aut}(\mathcal{M}(X, r, \alpha, \xi)) = \{ T = (\sigma, s, L, 0) \in \mathcal{T} \mid T(\xi) \cong \xi \} < \mathcal{T}$$

Theorem 4.8.2.

- The group of 3-birational transformations of the moduli space of parabolic vector bundle is

$$\text{Aut}_{3\text{-bir}}(\mathcal{M}(X, r, \alpha, \xi)) = \{ T \in \mathcal{T} \mid T(\xi) \cong \xi \} < \mathcal{T}$$

Corollary 4.7.11.

- Analysis on the stability space Δ of the moduli space of parabolic vector bundles

- Description of a numerical invariant $\overline{M}(r, \alpha, d) \in \mathbb{Z}^{\mathcal{N}}$ for some finite \mathcal{N} classifying univocally the numerical chambers.
- There exists a finite number of stability chambers (numerical or geometrical) in Δ . Proposition 4.9.3
- Characterization of geometrical chambers for high genus. Theorem 4.9.6, with genus bound refinements given by Proposition 4.9.7
- Computation of the automorphism group of the moduli space of framed bundles \mathcal{F} for a small choice of the stability parameter. The automorphisms are obtained as a composition of the following transformations
 - Taking the pullback with respect to an automorphism of the curve $\sigma : X \rightarrow X$ such that $\sigma(x) = x$.
 - Tensoring with a line bundle L on X
 - The canonical $\mathrm{PGL}_r(\mathbb{C})$ action on \mathcal{F} obtained by composing the framing with a linear automorphism $G : \mathbb{C}^r \rightarrow \mathbb{C}^r$

$$[G] \cdot (E, \alpha) = (E, G \circ \alpha)$$

In particular,

$$\mathrm{Aut}(\mathcal{F}) \cong \mathrm{PGL}_r(\mathbb{C}) \times \mathcal{T}$$

for a group \mathcal{T} fitting in the short exact sequence

$$1 \longrightarrow J(X)[r] \longrightarrow \mathcal{T} \longrightarrow \mathrm{Aut}(X, x) \longrightarrow 1.$$

Theorem 5.3.6 and Corollary 5.3.7.

Part of the results presented in this thesis have been summarized across several papers. I will also include here the main references of the corresponding articles in case the reader wants to access them independently

- The results in Chapter 3 about the moduli space of parabolic Λ -modules can be found in [Alf17].
- The proofs for the Torelli theorems of the moduli spaces of parabolic Higgs bundles, parabolic λ -connections and the parabolic Deligne–Hitchin moduli space (for arbitrary rank, determinant and generic weights) constitute a joint work with Tomás L. Gómez and are described in [AG18b] (see [AG16] for more references on Torelli type theorems for Deligne–Hitchin moduli spaces).
- The Torelli theorem for the moduli space of parabolic vector bundles (arbitrary rank, determinant and generic weights) and classification theorems for the isomorphisms and k -birational maps between moduli spaces of parabolic vector bundles described in Chapter 4 are a joint work with Tomás L. Gómez and are summarized in [AG18a]
- The analysis on the moduli space of framed bundles in Chapter 5 is a joint work with Indranil Biswas and it can be found in [AB18].

Chapter 2

Moduli spaces and their automorphism groups

In this chapter we will introduce some of the moduli spaces on which we will work through this thesis and review some results concerning their automorphisms. We will start by summarizing some of the existence theorems regarding moduli spaces of vector bundles with additional structures (Higgs bundles, connections, parabolic vector bundles, etc.).

Focusing on moduli spaces over curves, a natural question that arises when constructing these moduli spaces is whether the finally obtained scheme really depends on the algebraic structure of the curve. More precisely, when we construct a moduli space $\mathcal{M}(X)$ classifying some type of bundles on X , we can ask if there exist non-isomorphic X and X' such that $\mathcal{M}(X) \cong \mathcal{M}(X')$. This type of result is called a “Torelli type theorem” (in reference to the Torelli theorem for the Jacobian of a curve). We will review some of the main Torelli type theorems in the literature for the exposed moduli spaces.

Finally, we will focus on the main interest of this thesis: the computation of the symmetries of moduli spaces, i.e., the problem of classification of their automorphism groups. We will explore the state of the art on the subject for some of the moduli spaces introduced in the first part of the chapter.

2.1 Moduli spaces of vector bundles with additional structures

The moduli spaces of vector bundles enhanced with some extra structure (Higgs bundles, connections, Hitchin pairs, parabolic structures, etc.) have become almost omnipresent constructions in the actual landscape of algebraic geometry. They are related with many seemingly distant problems in areas such as representation theory, classification of solutions of PDEs on varieties or mathematical physics. Describing and extracting information about the geometry of these types of classification problems and their corresponding moduli spaces has therefore become an important task and, as a starting point, providing an algebraic construction for these moduli spaces is crucial. In this section we will review some of the existence results and works providing an algebraic construction for the main moduli spaces treated in the

rest of the thesis.

Starting from the moduli space of vector bundles, the first algebraic construction, as well as most of the development and the modern formalization of moduli theory, is due to Mumford. In [Mum62], he provided an algebraic construction for the moduli space of stable vector bundles $\mathcal{M}^s(X, r, d)$ using Geometric Invariant Theory (GIT). Its natural compactification – the moduli space of semistable vector bundles $\mathcal{M}(X, r, d)$ – was later on described by Seshadri [Ses67]. Moreover, he proved that the variety $\mathcal{M}^s(X, r, d) \subset \mathcal{M}(X, r, d)$ is non-singular. This result was refined by Narasimhan and Ramanan [NR75], who proved that if X is a smooth irreducible complete algebraic curve of genus $g \geq 2$, then the set of singular points of $\mathcal{M}(X, r, d)$ coincides with the locus of non-stable vector bundles, except for $g = 2$, $r = 2$ and even degree d .

Alternative GIT constructions for the moduli space of vector bundles were later on found by Gieseker [Gie77] and Maruyama [Mar77, Mar78].

Moving on to Higgs bundles, Hitchin described an analytic construction of the moduli space of rank two Higgs bundles on a Riemann surface in [Hit87]. An algebraic construction for the moduli space for arbitrary rank – but still over a curve – was described by Nitsure [Nit91]. Before the construction given by Nitsure, both the moduli space of Higgs bundles and the moduli space of connections were usually described analytically, as an infinite-dimensional symplectic reduction obtained from the gauge-theoretical formalism. On the other hand, Simpson [Sim94] developed the notion of Λ -module for a sheaf of rings of differential operator Λ as a way to study in a common framework different kinds of moduli spaces of vector bundles enhanced with some kind of “field” or “operator”, such as Higgs bundles, connections, logarithmic connections, etc. Inspired by a description made by Bernstein [Ber84] of the main properties of the sheaf of differential operators \mathcal{D}_X , Simpson compiled the notion of sheaf of rings of differential operators as a filtered sheaf of algebras satisfying some conditions that encapsulate and abstract the essential properties of the sheaf of differential operators \mathcal{D}_X . Then, Simpson gave a GIT construction of the moduli space of Λ -modules for any sheaf of rings of differential operators on a scheme of arbitrary dimension [Sim94]. As a particular case, GIT constructions for the moduli space of Higgs bundles (or, more generally, L -twisted Higgs bundles), connections or logarithmic connections can be found through his method.

Another moduli space whose construction can be achieved through the appropriate application of Simpson’s Λ -module framework is the moduli space of λ -connections, or Hodge moduli space. This moduli space is interesting, as it fibers over $\mathbb{A}_{\mathbb{C}}^1$ and “glues” together two non-isomorphic moduli spaces; the moduli space of Higgs bundles and the moduli space of connections. The generic fiber of the moduli space over any nonzero $\lambda \in \mathbb{C}$ is isomorphic to the moduli space of connections, but the fiber over $\lambda = 0$ is naturally identified with the moduli space of Higgs bundles. Therefore, the existence of this interpolating space proves that the moduli space of Higgs bundles can be obtained as a “degeneration” of the moduli space of connections. From Simpson–Corlette correspondence, we know that these two moduli spaces are diffeomorphic, although they are not isomorphic as algebraic varieties nor holomorphic varieties. As we will see in Section 3.9, the existence of this type of degenerating family can be used to prove that these two moduli spaces, despite not being biholomorphic, share some complex invariants. In particular, this

degenerating family has been used by Hausel and Thaddeus to prove that the moduli space of Higgs bundles and the moduli space of connections share the same stringy E-polynomials (polynomials whose coefficients are the stringy Hodge numbers).

Using this Hodge moduli space, Deligne [Del89] described a gluing construction of the twistor space of the moduli space of Higgs bundles called the Deligne–Hitchin moduli space. This space can be understood as a partial compactification of the Hodge moduli space. It is a holomorphic variety \mathcal{M}_{DH} fibrating over \mathbb{P}^1 such that

- The fiber over 0 is isomorphic to the Higgs moduli space of the curve X
- The fiber over a generic $\lambda \neq 0$ is isomorphic to the moduli space of connections on X (which is biholomorphic by the Riemann–Hilbert correspondence to the space of representations of $\pi_1(X)$)
- The fiber over $\lambda = \infty$ is isomorphic to the Higgs moduli space of the conjugated curve \bar{X} (the curve with the same differential structure but the opposite complex structure)

A similar construction was also used in [BGH13] to construct a Deligne–Hitchin moduli space for principal bundles with a semisimple structure group.

Now let us focus on the development of the parabolic versions of the previously presented moduli spaces. The moduli space of parabolic vector bundles over a curve was described by Mehta and Seshadri [MS80]. Maruyama and Yokogawa generalized the concept of parabolic sheaf to arbitrary dimension and proved the existence of a coarse moduli space of parabolic sheaves [MY92]. Later on, Yokogawa built the moduli space of parabolic Higgs bundles [Yok93].

It is worth mentioning that there is an important difference between the definition of parabolic structure given by Mehta and Seshadri and the one described by Maruyama and Yokogawa. In [MS80], a parabolic structure on a vector bundle E over a curved X with a set of marked points $D = \{x_1, \dots, x_n\}$ is presented as a filtration by subspaces of the fiber $E|_x$ over each parabolic point

$$E|_x = E_{x,1} \supsetneq E_{x,2} \supsetneq \dots \supsetneq E_{x,l_x} \supsetneq 0$$

together with real weights $0 \leq \alpha_1(x) < \dots < \alpha_{l_x}(x) < 1$. On the other hand, in [MS80], a parabolic structure on a sheaf E over a variety X with a marked divisor $D \subset X$ is presented as a single filtration

$$E = E_1 \supsetneq E_2 \supsetneq \dots \supsetneq E_l \supsetneq E_{l+1} = E(-D)$$

together with real weights $0 \leq \alpha_1 < \dots < \alpha_l < 1$. Therefore, while for Mehta and Seshadri the parabolic structures at different points (different components of the parabolic divisor D) are independent, in the formalism of Maruyama and Yokogawa, the parabolic structure is defined directly over the whole divisor D . Over a curve this does not make any significant difference, as the components of the divisor are disjoint (they are just different points), but on a higher dimensional variety the components $D_i \subset D$ generally intersect with each other and, in that case, providing a filtration on each restriction $E|_{D_i}$ is not equivalent to giving a filtration over the restriction of E to the whole divisor $E|_D$.

Concluding our review on moduli spaces of bundles with “singular” fields, we shall mention that the moduli space of logarithmic connections (without a parabolic structure) has been built by Nitsure [Nit93] and a notion of moduli space of parabolic connections was developed in [IIS06a] in order to study solutions to the Painlevé VI equation on \mathbb{P}^1 . There is a subtle difference between these two constructions. While both moduli spaces treat integrable connections over bundles E with a logarithmic singularity over some fixed divisor $D \subset X$, $\nabla : E \rightarrow E \otimes \Omega^1(\log(D))$, the moduli space built by Nitsure classifies pairs (E, ∇) consisting on a vector bundle over E and a logarithmic connection $\nabla : E \rightarrow E \otimes \Omega^1(\log(D))$, while the one considered by Inaba, Iwasaki and Saito parameterizes triples (E, E_\bullet, ∇) consisting on a parabolic vector bundle (E, E_\bullet) together with a logarithmic connection on E , $\nabla : E \rightarrow E \otimes \Omega^1(\log(D))$ such that the residue of the logarithmic connection at each parabolic point $x \in D$ preserves the filtration given by the parabolic structure

$$E|_x = E_{x,1} \supsetneq E_{x,2} \supsetneq \dots \supsetneq E_{x,l_x} \supsetneq 0$$

Observe that if the residue of a logarithmic connection $\text{Res}_x(\nabla) : E|_x \rightarrow E|_x$ has real eigenvalues then they are naturally ordered and then the fiber of the underlying vector bundle $E|_x$ is filtered naturally by the sum of the eigenspaces of increasingly large eigenvalues and, therefore, the logarithmic connection (E, ∇) has a natural structure of (quasi)parabolic connection (E, E_\bullet, ∇) . Nevertheless, the type of this filtration depends on the multiplicity of the eigenvalues and the actual Jordan decomposition of the residue. For instance, it is not always full flag. On the other hand, when we consider the moduli space of parabolic connections, we are classifying triples (E, E_\bullet, ∇) where (E, E_\bullet) has a predetermined type (e.g. full flag) and then we range over all possible connections which are compatible with that flag.

Finally, we will briefly talk about framed bundles. Recall that a frame bundle on a curve X with a framing over a point $x \in X$ is a vector bundle E together with a nonzero map $\alpha : E|_x \rightarrow \mathbb{C}^r$ called the framing. Notice that we have not required the map α to be an isomorphism, just to be nonzero. Framed bundles were first introduced by Donaldson as a tool to study the moduli space of instantons on \mathbb{R}^4 [Don84]. Later on, Huybrechts and Lehn [HL95a, HL95b] defined framed modules as a common generalization of several notions of decorated sheaves including framed bundles and Bradlow pairs. They described a general stability condition for framed modules and provided a GIT construction for the moduli space of framed modules.

2.2 Torelli type theorems

When treating a moduli problem and studying the corresponding moduli space, it is natural to question the dependence of the obtained scheme on the “parameters” used to define and construct it. In particular, one could analyze whether two instances of a moduli space with different choices of parameters could be isomorphic (or diffeomorphic, homeomorphic, birational, etc.). For example, the moduli space of curves obviously depends on the choice of a genus. The dimension of the moduli space of curves of genus $g > 1$ has dimension $3g - 3$, so moduli spaces for different genera are always non-isomorphic.

The moduli problems described through the last section are all based on the classification of bundles over a fixed variety X , possibly with a certain enhance-

ment (Higgs bundle, parabolic structure, etc.). In these cases the most prominent parameter of the moduli space is the variety X itself and we would like to know in which cases the isomorphism class of the moduli space identifies unequivocally the isomorphism class of X . For example, if X is a smooth complex projective curve of genus g , then the dimension of its Jacobian variety $\text{Jac}(X)$, is g , so the Jacobian of curves of different genera cannot be isomorphic. In fact, it is not enough to know the isomorphism class of $\text{Jac}(X)$ to recover the curve (there exist non-isomorphic curves with the same Jacobian). We need the additional data of its canonical polarization

Theorem 2.2.1 (Torelli's theorem). *Let X and X' be complete smooth curves of genus $g \geq 2$ over an algebraically closed field k . Let θ_X and $\theta_{X'}$ be the canonical polarizations of $\text{Jac}(X)$ and $\text{Jac}(X')$ respectively. If $(\text{Jac}(X), \theta_X)$ is isomorphic to $(\text{Jac}(X'), \theta_{X'})$ as polarized varieties over k then X and X' are isomorphic over k .*

In the literature, this is known as the Torelli theorem. A proof of the theorem can be found in [CS86, Corollary §7.12.2]. In general, if we consider a moduli space $\mathcal{M}(X)$ classifying some kind of geometric objects over a variety X (vector bundles, Higgs bundles, etc.), we will call a “Torelli type theorem” any result stating that the isomorphism class of the moduli space $\mathcal{M}(X)$ uniquely determines the isomorphism class of X .

Torelli type theorems are not restricted to curves. For example, [LP80] and [Fri84] proved a Torelli type result for K3 surfaces. However, in this thesis we will restrict ourselves to the analysis of moduli spaces over curves. In this context, the natural generalization for the Torelli theorem arises by passing from the moduli space of line bundles over X to the moduli space of rank r vector bundles over X . Mumford and Newstead [MN68] proved the following Torelli theorem for the moduli space of stable rank 2 vector bundles with fixed determinant of odd degree.

Theorem 2.2.2 (Torelli vector bundles [MN68]). *Let X and X' be smooth complex curves of genus $g \geq 2$. Let ξ and ξ' be line bundles of odd degree over X and X' respectively. Let $\mathcal{M}(2, \xi, X)$ and $\mathcal{M}(2, \xi', X')$ be the moduli spaces of stable vector bundles of rank 2 and determinant ξ or ξ' respectively. If $\mathcal{M}(2, \xi, X) \cong \mathcal{M}(2, \xi', X')$ then $X \cong X'$.*

Later on, Tyurin [Tyu70] extended this result to arbitrary rank r when $\deg(\xi)$ is coprime to r . An alternative proof was also given by Narasimhan and Ramanan [NR75], showing that the intermediate Jacobian of $\mathcal{M}(X, r, \xi)$ with the induced polarization by $\mathcal{M}(X, r, \xi)$ is isomorphic to $\text{Jac}(X)$ as polarized varieties. Then we can apply the classical Torelli theorem to complete the proof. Finally, Kouvidakis and Pantev [KP95] proved the result for arbitrary rank and degree as part of their work on the computation of the automorphism group of the moduli space of vector bundles.

Theorem 2.2.3 (Torelli vector bundles [KP95]). *Let X and X' be smooth complex curves of genus $g \geq 3$. Let ξ and ξ' be line bundles of degree d on X and X' respectively. If $\mathcal{M}(r, \xi, X) \cong \mathcal{M}(r, \xi', X')$ then $X \cong X'$.*

In this case, the idea of the proof is to study the abelianization of the isomorphism $\mathcal{M}(X, r, \xi) \rightarrow \mathcal{M}(X', r, \xi')$ by lifting it to a map between the cotangent

bundles $T^*\mathcal{M}(X, r, \xi) \rightarrow T^*\mathcal{M}(X', r', \xi')$ and then restricting it to a map between the corresponding Prym varieties through the spectral construction

$$\begin{array}{ccc} \mathrm{Prym}(\tilde{X}, X) & \longrightarrow & \mathrm{Prym}(\tilde{X}', X') \\ \downarrow & & \downarrow \\ W_{\mathrm{reg}} & \longrightarrow & W'_{\mathrm{reg}} \end{array}$$

where W_{reg} and W'_{reg} are the subset of points of the Hitchin spaces of X and X' respectively corresponding to nonsingular spectral curves (see 4.2 and 4.3 for definitions) and \tilde{X} and \tilde{X}' are the universal spectral curves over $W_{\mathrm{reg}} \times X$ and $W'_{\mathrm{reg}} \times X'$. Other different proofs have appeared in the literature deepening in this idea of using the geometry of the Hitchin map to obtain information about the structure of the automorphisms and the isomorphism class of $\mathcal{M}(X, r, \xi)$. Hwang and Ramanan [HR04] gave an alternative proof of this result proving that the Hitchin discriminant (the locus of singular spectral curves) can be geometrically identified inside $T^*\mathcal{M}(X, r, \xi)$ through the analysis of the Hecke curves in $\mathcal{M}(X, r, \xi)$. Then, they proved that the curve X is the dual variety of some part of the image of the discriminant.

This idea was further explored and simplified by Biswas and Gómez in [BGM13], as they proved that the Hitchin discriminant can be identified with the union of the complete rational curves in $T^*\mathcal{M}(X, r, \xi)$.

On the other hand, a Torelli type theorem for the moduli space of Higgs bundles was proven by Biswas and Gómez in [BG03], under the assumption of coprimality between the rank and the degree. Their proof was later on simplified in [BGHL09] and generalized to principal bundles with semisimple structure groups in [BGH13], extending the result to arbitrary rank and degree in the particular case of vector bundles.

Theorem 2.2.4 (Torelli Higgs bundles [BG03, BGHL09, BGH13]). *Let X and X' be compact connected Riemann surfaces of genus at least 3 and let ξ and ξ' be line bundles over X and X' respectively. Let $\mathcal{M}_{\mathrm{Higgs}}(X, r, \xi)$ be the moduli space of semistable Higgs bundles on X with rank r and determinant ξ . If $\mathcal{M}_{\mathrm{Higgs}}(X, r, \xi) \cong \mathcal{M}_{\mathrm{Higgs}}(X', r, \xi')$ is a biholomorphism then $X \cong X'$.*

In their proof, they make use of the fact that the moduli space of vector bundles is naturally immersed inside the moduli space of Higgs bundles, as

$$\mathcal{M}(X, r, \xi) \subset T^*\mathcal{M}(X, r, \xi) \subset \mathcal{M}_{\mathrm{Higgs}}(X, r, \xi)$$

Biswas and Gómez prove that this subset can be geometrically characterized through the analysis of the fix point locus of the \mathbb{C}^* -actions on the moduli space $\mathcal{M}_{\mathrm{Higgs}}(X, r, \xi)$.

This idea was extended in [BGHL09] to prove two additional Torelli type results for the Hodge moduli space and the Deligne–Hitchin moduli space over a curve, using the fact that $\mathcal{M}(X, r, \xi)$ is canonically embedded into these spaces in the following way

$$\mathcal{M}(X, r, \mathcal{O}_X) \subset T^*\mathcal{M}(X, r, \mathcal{O}_X) \subset \mathcal{M}_{\mathrm{Higgs}}(X, r, \mathcal{O}_X) \subset \mathcal{M}_{\mathrm{Hod}}(X, r) \subset \mathcal{M}_{\mathrm{DH}}(X, r)$$

The case of the Deligne–Hitchin moduli space is slightly different from the other results, as the curve X cannot be identify uniquely from the isomorphism class of

$\mathcal{M}_{\text{DH}}(X, r)$. Observe from our description of the moduli space that if \overline{X} is the conjugate curve of X (i.e., the curve with the opposite complex structure), then we have a canonical identification $\mathcal{M}_{\text{DH}}(X, r) \cong \mathcal{M}_{\text{DH}}(\overline{X}, r)$. Therefore, we will never be able to distinguish a curve X and its conjugate \overline{X} from the geometry of their Deligne–Hitchin moduli space. Instead, we obtain the following result

Theorem 2.2.5 (Torelli Deligne–Hitchin, [BGHL09]). *Let X and X' be compact connected Riemann surfaces of genus $g \geq 3$. If $\mathcal{M}_{\text{DH}}(X, r) \cong \mathcal{M}_{\text{DH}}(X', r)$ is a biholomorphism then $X' \cong X$ or $X' \cong \overline{X}$.*

Moreover, this theorem also holds for principal bundles with semisimple structure group [BGH13].

2.2.1 Torelli theorems for moduli spaces of parabolic bundles

Moving to the parabolic scenario, until recently the main Torelli type theorem known for the moduli space of parabolic vector bundles was the one given by Balaji, del Baño and Biswas [BdBnB01]. Let X be a compact connected Riemann surface and let D be a finite set of points in X . We will call a rank 2 full flag system of weights α over (X, D) “small” if

$$\sum_{x \in D} (\alpha_1(x) + \alpha_2(x)) < 1$$

Theorem 2.2.6 (Torelli parabolic vector bundles [BdBnB01]). *Let X and X' be compact connected Riemann surfaces of genus at least 2 and let ξ and ξ' be line bundles of degree 1 over X and X' respectively. Let D and D' be finite sets of points over X and X' respectively, and let α and α' be “small” rank 2 full flag systems of weights over (X, D) and (X', D') respectively. If $\mathcal{M}(X, D, 2, \alpha, \xi) \cong \mathcal{M}(X', D', 2, \alpha', \xi')$ then there exists an isomorphism $X \cong X'$ sending D to D' .*

Observe that in the parabolic case the expected Torelli type results like the previous one recover more than the isomorphism class of the curve X ; they recover the curve X together with the set of marked points $D \subset X$. This does not mean that the isomorphism $\sigma : X \xrightarrow{\sim} X'$ described by the theorem preserves every single parabolic point, but it must send the divisor $D \subset X$ to $D' \subset X'$, i.e., $\sigma(D) = D'$. The “small weights” condition is chosen in [BdBnB01] so that for every parabolic vector bundle (E, E_\bullet) of rank 2 and degree 1 on (X, D) with system of weights α , (E, E_\bullet) is stable as a parabolic vector bundle if and only if E is stable as a vector bundle. Therefore, the moduli space $\mathcal{M}(X, D, 2, \alpha, \xi)$ admits a forgetful map to the moduli space of rank 2 stable bundles with determinant ξ , $\mathcal{M}(X, 2, \xi)$. In fact, if \mathcal{E} is the universal bundle over the moduli space $\mathcal{M}(X, 2, \xi)$, then if $D = x_1 + \dots + x_n$

$$\mathcal{M}(X, D, 2, \alpha, \xi) \cong \mathbb{P}(\mathcal{E}|_{\{x_1\} \times \mathcal{M}(X, 2, \xi)}) \times_{\mathcal{M}(X, 2, \xi)} \cdots \times_{\mathcal{M}(X, 2, \xi)} \mathbb{P}(\mathcal{E}|_{\{x_n\} \times \mathcal{M}(X, 2, \xi)})$$

The authors exploit this structure to obtain an explicit generator for each boundary line of the numerically effective cone of $\mathcal{M}(X, D, 2, \alpha, \xi)$. Each one of these generators gives a map from $\mathcal{M}(X, D, 2, \alpha, \xi)$ to a certain moduli space of vector bundles $\mathcal{M}(X, 2, \xi_0)$ for some line bundle ξ_0 . In particular, the pullback of the determinant bundle on $\mathcal{M}(X, 2, \xi)$ by the forgetful map gives one such generator, and it is the

only one giving rise to a map to a smooth variety if $g \geq 3$ (the others induce maps to moduli spaces with $\deg(\xi) = 0$, and are therefore singular for $g \geq 3$). Therefore, one can geometrically identify the boundary line of the numerically effective cone of $\mathcal{M}(X, D, 2, \alpha, \xi)$ generated by the pullback of the determinant bundle as the only component such that if we take a sufficiently divisible nontrivial line bundle in the half-line, then it induces a map to a smooth variety. Thus, we recover the forgetful map $\mathcal{M}(X, D, 2, \alpha, \xi) \rightarrow \mathcal{M}(X, 2, \xi)$ and we can recover the isomorphism class of X through any of the Torelli type theorems for vector bundles previously described (for example, Theorem 2.2.2, proved in [MN68]). The parabolic points are recovered by studying the map $X \rightarrow H_{\mathcal{D}}^2(\mathcal{M}(X, 2, \xi), \mathbb{Z}(2))$ sending a point $x \in X$ to the cohomology class $c_2(\mathcal{E}|_{\{x\} \times \mathcal{M}(X, 2, \xi)})$ in the Deligne-Beilinson cohomology of $\mathcal{M}(X, 2, \xi)$. They proved that this map sends the orbit of the automorphism group of the curve $\text{Aut}(X)$ to the orbit of the automorphism group of the moduli space $\text{Aut}(\mathcal{M}(X, 2, \xi))$. From the structure of the boundary lines of the numerically effective cone, we can recover the bundles $\mathbb{P}(SE|_{\{x\} \times \mathcal{M}(X, 2, \xi)}) \rightarrow \mathcal{M}(X, 2, \xi)$ up to an automorphism of $\mathcal{M}(X, 2, \xi)$. Lifting the projective bundle to a suitable vector bundle representative, we can compute the class $c_2(SE|_{\{x\} \times \mathcal{M}(X, 2, \xi)})$ up to the action of an automorphism of $\text{Aut}(\mathcal{M}(X, r, \xi))$ and, therefore, we recover $x \in X$ up to an action of $\text{Aut}(X)$.

In [Seb11], Sebastian slightly extended this proof to the case where the rank is arbitrary, but the parabolic system is still of length 2 (i.e., we still give two parameters per point), and corresponds to the choice of a hyperplane inside each fiber. Under these conditions on the parabolic type and a similar notion of “small” parameters α , the moduli space is still isomorphic to

$$\mathcal{M}(X, D, r, \alpha, \xi) \cong \mathbb{P}(\mathcal{E}|_{\{x_1\} \times \mathcal{M}(X, 2, \xi)}) \times_{\mathcal{M}(X, 2, \xi)} \cdots \times_{\mathcal{M}(X, 2, \xi)} \mathbb{P}(\mathcal{E}|_{\{x_n\} \times \mathcal{M}(X, 2, \xi)})$$

and a similar analysis to the one used in [BdBnB01] can be applied.

All the previous results deal with the moduli spaces of bundles with fixed determinant. Recently, Biswas, Gómez and Logares [BGL16] proved a generalization of the Torelli theorem for the moduli space of parabolic vector bundles that worked on the non-fixed determinant situation for arbitrary rank and degree. Moreover, the result holds for generic parabolic weights. Let $\mathcal{M} = \mathcal{M}(X, D, r, \alpha, d)$ denote the moduli space of stable parabolic vector bundles of rank r , degree d and system of weights α over (X, D) . We can construct a natural determinant bundle \mathcal{L} over \mathcal{M} in the following way. Let $\chi = \chi(E)$ be the Euler characteristic of any (and therefore all) underlying vector bundles of points in $\mathcal{M}(X, D, r)$ (it only depends on r and d , so it is fixed). Let $p : X \times \mathcal{M} \rightarrow \mathcal{M}$ be the canonical projection and let \mathcal{E} be the universal bundle over $X \times \mathcal{M}$. Fix any point $x \in X$. Then

$$\mathcal{L} = \det(Rq_* \mathcal{E})^{-r} \otimes (\wedge^r \mathcal{E}|_{\{x\} \times \mathcal{M}})^{\chi}$$

is a line bundle over $X \times \mathcal{M}$ whose fiber over each parabolic vector bundle (E, E_{\bullet}) can be canonically identified with $\det(E)$. We will call it the determinant bundle. Observe that this line bundle exists even if the universal bundle \mathcal{E} does not (if the terms of the parabolic type, rank and degree have all a common divisor greater than one). Then \mathcal{L} gives a natural polarization of \mathcal{M} .

Theorem 2.2.7 (Torelli theorem for parabolic vector bundles [BGL16]). *Let X and X' be smooth complex projective curves of genus $g \geq 4$ with parabolic points D and D' respectively. Let α and α' be generic systems of weights over (X, D) and (X', D') respectively. Let \mathcal{L} and \mathcal{L}' denote the determinant bundles over $\mathcal{M}(X, D, r, \alpha, d)$ and $\mathcal{M}(X', D', r, \alpha', d)$ respectively. If there is an isomorphism $\varphi : \mathcal{M}(X, D, r, \alpha, d) \rightarrow \mathcal{M}(X', D', r, \alpha', d)$ such that $\varphi^* \text{NS}(\mathcal{L}') = \text{NS}(\mathcal{L})$, then $(X, D) \cong (X', D')$.*

Notice that, contrary to what we have seen in the rank 2 version of the Torelli theorem in [BdBnB01], in this version of the Torelli theorem it is mandatory to provide a canonical polarization together with the moduli space and to ask the isomorphism to preserve the Neron-Severi class of the polarization. This kind of requirement mirrors the one appearing in the classical Torelli theorem for the Jacobian of a curve. Nevertheless, at the end of Chapter 4 we will prove a generalization of this result that demonstrates that if the weights are full flag then we do not need to preserve the polarization to recover the isomorphism class of the marked curve, the isomorphism class of the moduli space $\mathcal{M}(X, r, \alpha, \xi)$ is enough to identify unequivocally the isomorphism class of the marked curve (Theorem 4.3.6). In fact, we will provide an ever more refined version of this theorem, as we will demonstrate that the 3-birational equivalence class of the moduli space (equivalence class of schemes which admit a birational equivalence with $\mathcal{M}(X, r, \alpha, \xi)$ which restricts to an isomorphism on a “big” dense subset) determines uniquely the isomorphism class of the marked curve.

Analogously to what happens for the moduli space of vector bundles, the Torelli theorem for the moduli space of parabolic vector bundles can be used to prove other Torelli type theorems for moduli spaces of parabolic vector bundles with some additional structure, such as the moduli space of parabolic Higgs bundles or the parabolic Hodge moduli space. The common idea behind these proofs is the same as in the non-parabolic case; if we manage to characterize geometrically a subset of a moduli space which is isomorphic to a moduli space of parabolic vector bundles then we use the Torelli theorem for the latter to recover the isomorphism class of the marked curve.

One of the main results on parabolic moduli spaces relying on this idea is the Torelli theorem for the moduli space of parabolic Higgs bundles proved by Gómez and Logares [GL11]. Given a marked curve (X, D) , a line bundle ξ over X and a system of weights α over (X, D) , let $\mathcal{M}_{\text{Higgs}}(X, r, \alpha, \xi)$ be the moduli space of (strongly) parabolic Higgs bundles over X with fixed determinant ξ , parabolic weights α and rank r .

Theorem 2.2.8 (Torelli theorem for parabolic Higgs bundles [GL11]). *Let X and X' be smooth complex projective curves of genus $g \geq 2$ with marked points $D = \{x_1, \dots, x_n\} \subset X$ and $D' = \{x'_1, \dots, x'_n\} \subset X'$. Let ξ and ξ' be line bundles over X and X' of degree 1 and let α and α' be full flag small systems of weights. If $\mathcal{M}_{\text{Higgs}}(X, r, \alpha, \xi) \cong \mathcal{M}_{\text{Higgs}}(X', 2, \alpha', \xi')$ then $(X, D) \cong (X', D')$.*

The idea of the proof is similar to the one used in Theorem 2.2.4. The moduli space of parabolic vector bundles $\mathcal{M}(X, r, \alpha, \xi)$ is immersed as the zero section of its cotangent bundle which is a dense open subset of the moduli space of (strongly)

parabolic Higgs bundles

$$\mathcal{M}(X, r, \alpha, \xi) \subset T^*\mathcal{M}(X, r, \alpha, \xi) \subset \mathcal{M}_{\text{Higgs}}(X, r, \alpha, \xi)$$

Gómez and Logares characterized this subset geometrically analyzing the \mathbb{C}^* actions on the moduli space, and then apply Theorem 2.2.6 to recover the isomorphism class of the marked curve (X, D) . Nevertheless, relying on the Torelli result in [BdBnB01] implies automatically that we must restrict ourselves to rank 2, degree 1 and small systems of weights.

Conversely, the rest of the proof of the Theorem 2.2.8 holds with more generality. Although the assumptions of having concentrated weights and coprime rank r and degree d are still necessary for their proof, Gómez and Logares provided a geometric characterization for the subset $\mathcal{M}(X, r, \alpha, \xi) \subset \mathcal{M}_{\text{Higgs}}(X, r, \alpha, \xi)$ for arbitrary rank. Thus, if Torelli Theorem 2.2.6 was extended to higher rank then the results in [GL11] would directly be generalized to higher rank, coprime degree and small full flag systems of weights.

Later on, in [AG18b], we used once again the ideas from [GL11] of using the \mathbb{C}^* -actions on the moduli space $\mathcal{M}_{\text{Higgs}}(X, r, \alpha, \xi)$ to characterize the subset $\mathcal{M}(X, r, \alpha, \xi) \subset \mathcal{M}_{\text{Higgs}}(X, r, \alpha, \xi)$ and we refined Gómez and Logares' proof so that we could drop both the assumption of coprimality between the rank and the degree and the need of small systems of weights. Unfortunately, even with these improvements, at the moment the resulting statement of the Torelli theorem was exactly the same as the one given in Theorem 2.2.8 because, similarly to [GL11], we relied on the application of Theorem 2.2.6 by [BdBnB01] to complete the proof. Thus, even if the rest of the construction was suitable for generic full flag weights and arbitrary rank and determinant, for the last step we still needed to assume rank 2, degree 1 and small weights in order to apply the Torelli theorem in [BdBnB01]. As a slight improvement, instead of requiring “small” weights, we proved that the actual numerical condition required on the parabolic weights to guarantee the equivalence between the stability of a parabolic vector bundle and the stability of its underlying vector bundle needed through the proof in [BdBnB01] was to ask the weights to be “concentrated”. Roughly speaking, from a stability point of view we do not really care if the weights are big or small, but rather if the difference between the first and the last weights over each point $\alpha_r(x) - \alpha_1(x)$ is small enough as adding a constant small $\varepsilon \in \mathbb{R}$ to each weight $\alpha_i(x) \mapsto \alpha_i(x) + \varepsilon$ does not change the stability of the parabolic vector bundles.

In any case, the dependence on Theorem 2.2.6 represents a bottleneck for the results in [GL11] and [AG18b]. In this thesis (Section 4.3) we will provide a solution to this issue, as Theorem 4.3.6 provides a suitable substitute for the results in [BdBnB01] which overpasses the restrictions on the rank, the degree and the need of small/concentrated weights.

Moreover, the techniques in [AG18b] are not restricted to recovering the moduli space of parabolic vector bundles inside the moduli space of parabolic Higgs bundles. Instead, we are able to characterize geometrically the following natural chain of immersions of moduli spaces

$$\mathcal{M}(X, r, \alpha, \xi) (\subset T^*\mathcal{M}(X, r, \alpha, \xi)) \subset \mathcal{M}_{\text{Higgs}}(X, r, \alpha, \xi) \subset \mathcal{M}_{\text{Hod}}(X, r, \alpha, \xi) \subset \mathcal{M}_{\text{DH}}(X, r, \alpha, \xi)$$

where $\mathcal{M}_{\text{Hod}}(X, r, \alpha, \xi)$ is the parabolic Hodge moduli space, parameterizing parabolic λ -connections on (X, D) and $\mathcal{M}_{\text{DH}}(X, r, \alpha, \xi)$ is the parabolic analogue of the Deligne–Hitchin moduli space. In the last case, it is worth to mention that the isomorphism class of the parabolic Deligne–Hitchin moduli space is not enough to recover the isomorphism class of the marked curve. As it is built gluing together the moduli spaces of Hodge bundles for a marked curve (X, D) and the complex curve (\bar{X}, \bar{D}) obtained taking the opposite complex structure on X , the parabolic Deligne–Hitchin moduli space for (X, D) and for its complex conjugate (\bar{X}, \bar{D}) is exactly the same, i.e. these two complex structures are indistinguishable from the point of view of the moduli space. Instead, the Torelli type theorem for this case implies that we can recover the set $\{(X, D), (\bar{X}, \bar{D})\}$ consisting on the marked curve and its conjugate.

Therefore, combining this characterization with the Torelli theorem for parabolic vector bundles we obtain a Torelli type theorem for the following moduli spaces

- Parabolic Higgs moduli space $\mathcal{M}_{\text{Higgs}}(X, r, \alpha, \xi)$
- Parabolic Hodge moduli space $\mathcal{M}_{\text{Hod}}(X, r, \alpha, \xi)$
- Parabolic Deligne–Hitchin moduli space $\mathcal{M}_{\text{DH}}(X, r, \alpha)$

Nevertheless, as it happened in [GL11], the dependence on the Torelli result in [BdBnB01] forces us to restrict the final result to the case of rank 2, degree 1 and concentrated weights. Once again, the generalization of this theorem provided by Theorem 4.3.6 unlocks analogous results for arbitrary rank, degree and generic parabolic weights.

2.2.2 Torelli for the moduli space of framed bundles

For the case of the moduli space of framed bundles, a Torelli type theorem was developed by Biswas, Gómez and Muñoz[BGM10]. Similarly to parabolic vector bundles, the moduli space of framed bundles depends on a stability parameter $\tau \in \mathbb{R}$. If the parameter is “small” enough (in the sense that $\tau < \tau(r)$ for some specific bound depending on the rank of the bundle), the τ -semistability of a framed bundle is equivalent to the semistability of the underlying vector bundle. Under this conditions, the moduli space of framed bundles is the total space of a projective bundle on the moduli space of vector bundles. In [BGM10] this case is studied and the following Torelli type theorem is proven

Theorem 2.2.9 (Torelli for framed bundles [BGM10]). *Let X be a smooth projective curve of genus $g \geq 2$ and $x \in X$ be a point. Let $r \geq 2$ be an integer and let ξ be a line bundle over X . Assume that $g > 2$ if $r = 2$. Let $\tau \in \mathbb{R}$ with $0 < \tau < \tau(r)$. Then let $\mathcal{F}(X, x, r, \tau, \xi)$ be the moduli space of τ -semistable framed bundles over (X, x) with rank r and determinant ξ . Let $X', g', x' \in X', r', \xi'$ and τ' be another set of data satisfying the analogous conditions. If $\mathcal{F}(X, x, r, \tau, \xi) \cong \mathcal{F}(X', x', r', \tau', \xi')$ then $r = r'$ and there is an isomorphism $\sigma : X \rightarrow X'$ sending $\sigma(x) = x'$.*

Under the prescribed conditions by the theorem, the moduli space $\mathcal{F} = \mathcal{F}(X, x, r, \tau, \xi)$ admits a natural $\text{PGL}_r(\mathbb{C})$ -action in the following way. If (E, α) is a framing with

$\alpha : E|_x \rightarrow \mathbb{C}^r$, then given any matrix $[G] \in \mathrm{PGL}_r(\mathbb{C})$ we can compose α with the automorphism $G : \mathbb{C}^r \rightarrow \mathbb{C}^r$ to obtain a new framing for E

$$(E, \alpha) \mapsto (E, G \circ \alpha)$$

It can be proved that this transformation preserves stability and, therefore, defines an action $\mathrm{PGL}_r(\mathbb{C}) \times \mathcal{F} \rightarrow \mathcal{F}$. The key idea of the proof in [BGM10] is that this action is essentially the unique nontrivial $\mathrm{PGL}_r(\mathbb{C})$ action on \mathcal{F} . Then they prove that the GIT quotient of $\mathcal{F}(X, x, r, \tau, \xi)$ by this action for any linearized polarization coincides with the moduli space $\mathcal{M}(X, r, \xi)$ of semistable vector bundles of rank r and determinant ξ over X . Moreover, the canonical projection from the semistable part of \mathcal{F} to the GIT quotient coincides precisely with the forgetful map. This way, they can reduce the problem to the Torelli theorem for parabolic vector bundles.

This idea of recovering canonically the $\mathrm{PGL}_r(\mathbb{C})$ action on \mathcal{F} will be of great relevance for our posterior analysis on the automorphisms of \mathcal{F} , developed in Chapter 5.

2.3 Symmetries and automorphisms of moduli spaces

From the classical Torelli theorem for the Jacobian variety, we know that if $\Phi : J(X) \rightarrow J(X')$ is an isomorphism between the Jacobian varieties of two curves X and X' which respects the canonical polarization $\Phi^* \theta_{X'} = \theta_X$, then there must be an isomorphism $\sigma : X \xrightarrow{\sim} X'$. Clearly, given an isomorphism $\sigma : X \rightarrow X'$, the pullback map induces an isomorphism $\sigma^* J(X') \rightarrow J(X)$ which preserves the polarization. Nevertheless, in general this is not the only admissible isomorphism between these two varieties, i.e., they are not canonically isomorphic.

Clearly, if $\Phi, \Psi : J(X) \rightarrow J(X')$ are two isomorphisms between the polarized Jacobians then $\Phi^{-1} \circ \Psi : J(X) \rightarrow J(X)$ is an automorphism of $(J(X), \theta_X)$, so in order to understand the possible isomorphisms between Jacobian varieties it is enough to understand the automorphism group of a polarized Jacobian.

As $J(X)$ is abelian, then each automorphism $\Phi : J(X) \rightarrow J(X)$ decomposes as $\Phi = T_L \circ \Phi_0$ (c.f. [Mil08, Corollary 1.2]), where

- $T_L : J(X) \rightarrow J(X)$ is the translation by a degree zero line bundle L , sending $\xi \mapsto \xi \otimes L$
- $\Phi_0 : J(X) \rightarrow J(X)$ is an homomorphism of abelian varieties, i.e., an automorphism of $J(X)$ preserving the group structure.

the original isomorphism $\Phi^{-1} : J(X') \rightarrow J(X)$ may not coincide with $\sigma^* : J(X') \rightarrow J(X)$. In particular, it is straightforward to prove that

$$\mathrm{Aut}(J(X)) \cong J(X) \rtimes \mathrm{Aut}_{\mathrm{grp}}(J(X))$$

and when we consider automorphisms of the polarized Jacobian

$$\mathrm{Aut}((X), \theta_X) \cong J(X) \rtimes \mathrm{Aut}_{\mathrm{grp}}((J(X), \theta_X))$$

Then, combining this decomposition with the Torelli result leads to the following well known description of the automorphism group of the Jacobian which is usually known as the “strong” Torelli theorem

Theorem 2.3.1 (Strong Torelli theorem).

$$\mathrm{Aut}((J(X), \theta_X)) \cong \begin{cases} J(X) \rtimes (\mathbb{Z}/2\mathbb{Z} \times \mathrm{Aut}(X)) & X \text{ non-hyperelliptic} \\ J(X) \rtimes \mathrm{Aut}(X) & X \text{ hyperelliptic} \end{cases}$$

In this decomposition, the $\mathbb{Z}/2\mathbb{Z}$ factor corresponds to the dualization of a line bundle, i.e, the map $(\cdot)^\vee : J(X) \rightarrow J(X)$ sending $\xi \mapsto \xi^{-1}$. If the curve is hyperelliptic this type of transformation can be expressed in terms of a pullback by the hyperelliptic involution, so it does not generate a new automorphism of the Jacobian (there is a non-injective map $J(X) \rtimes (\mathbb{Z}/2\mathbb{Z} \times \mathrm{Aut}(X)) \rightarrow \mathrm{Aut}((J(X), \theta_X))$).

As we mentioned in the introduction, the generalization of this result to higher rank was performed by Kouvidakis and Pantev [KP95]. They proved that the automorphism group of the moduli space of vector bundles on X with rank r and fixed determinant ξ , $\mathcal{M}(X, r, \xi)$ is generated by suitable combinations of the following three types of transformations

- Taking the pullback by an automorphism of the curve $\sigma : X \rightarrow X$
- Dualizing $E \mapsto E^\vee$
- Tensoring with a line bundle over X , $E \mapsto E \otimes L$

Clearly, these three transformations might change the determinant of the bundle E , but they preserve the stability. In order to describe the actual automorphisms of $\mathcal{M}(X, r, \xi)$ we need to consider combinations of these transformations that actually preserve the determinant. Let us consider the following subgroups $\mathcal{G}_\xi^+ \subset \mathcal{G}_\xi \subset J(X) \times \mathrm{Aut}(X)$

$$\begin{aligned} \mathcal{G}_\xi &= \{(L, \sigma) \mid L^r = \xi \otimes \sigma^* \xi^{\pm 1}\} \\ \mathcal{G}_\xi^+ &= \{(L, \sigma) \mid L^r = \xi \otimes \sigma^* \xi^{-1}\} \end{aligned}$$

The subgroup \mathcal{G}_ξ^+ is clearly normal and, if r does not divide $2 \deg(\xi)$, then $\mathcal{G}_\xi^+ = \mathcal{G}_\xi$. Otherwise, it is a proper subgroup of index 2. Then Kouvidakis and Pantev prove the following result

Theorem 2.3.2 (Automorphisms moduli of vector bundles [KP95]). *Let X be a smooth curve of genus $g \geq 3$. Then the map*

$$\begin{aligned} \mathcal{G}_\xi &\longrightarrow \mathrm{Aut}(\mathcal{M}(X, r, \xi)) \\ (L, \sigma) &\longmapsto \left(E \mapsto \begin{cases} \sigma^* E \otimes L & \text{if } (L, \sigma) \in \mathcal{G}_\xi^+ \\ \sigma^* E^\vee \otimes L & \text{if } (L, \sigma) \in \mathcal{G}_\xi \setminus \mathcal{G}_\xi^+ \end{cases} \right) \end{aligned}$$

is an isomorphism.

This classification theorem was also proved by Hwang and Ramanan [HR04] based on the analysis of the Hecke curves in the moduli space mentioned in the last section. Later on, their proof was further simplified by Biswas, Gómez and Muñoz [BGM13]. Moreover, the techniques in [BGM13] have also been applied to obtain similar classification results for the moduli space of symplectic bundles [BGM12] and the moduli spaces of principal bundles with structure group F_4 or E_6 [Sán18].

Finally, I shall mention that the automorphism group of the moduli space of Higgs bundles $\mathcal{M}_{\mathrm{Higgs}}(X, r, \xi)$ has been computed by Baraglia [Bar16]. He proved that the automorphisms of $\mathcal{M}_{\mathrm{Higgs}}(X, r, \xi)$ are generated by

- Automorphisms of the moduli space of vector bundles $\mathcal{M}(X, r, \xi)$ (i.e., pull-back, tensorization and dualization)
- The canonical \mathbb{C}^* -action $(E, \Phi) \mapsto (E, \lambda\Phi)$
- Vertical flows of the Hitchin system

Chapter 3

Moduli space of parabolic Λ -modules over a curve

Simpson [Sim94] developed the concept of Λ -modules as a theoretical framework that unified the notions of vector bundle, Higgs bundle, integrable connection and other similar geometric structures. The main idea is to consider the corresponding Higgs field or connection as an action of a certain sheaf of rings of differential operators on a coherent sheaf. For example, if we have a Higgs field $\varphi : E \rightarrow E \otimes K$ over a coherent sheaf E with $\varphi \wedge \varphi = 0$, it induces a morphism $\varphi' : K^\vee \otimes E \rightarrow E$ that extends, by composition, to a morphism $\varphi'' : \mathrm{Sym}^\bullet(K^\vee) \otimes E \rightarrow E$. Therefore, providing a Higgs field is equivalent to defining a left action of the sheaf of algebras $\Lambda^{\mathrm{Higgs}} := \mathrm{Sym}^\bullet(K^\vee)$ on E .

Similarly, sheaves with an integrable connection, described as a sheaf E together with a \mathbb{C} -linear morphism $\nabla : E \rightarrow E \otimes K$ satisfying the Leibniz rule such that $\nabla^2 = 0$, are in correspondence with \mathcal{D}_X -modules, i.e., sheaves E with a left action of the sheaf of differential operators $\Lambda^{\mathrm{DR}} := \mathcal{D}_X$. This approach had been studied by [Ber84] and motivated the definition given by Simpson of sheaf of rings of differential operators. A sheaf of rings of differential operators over X is a filtered \mathcal{O}_X -algebras satisfying some conditions resembling the main properties of \mathcal{D}_X ; the left and right action of \mathcal{O}_X on the graduate are the same (the algebra of symbols of operators of a certain degree is commutative), the graduate at each point is coherent (the algebra of symbols of operators of a given order is finite-dimensional) and the graded algebra is generated by the first step of the filtration (the algebra of differential operators is generated by operators of order one).

A Λ -module is a left module E for the sheaf of rings Λ where the \mathcal{O}_X -module structure coming from $\mathcal{O}_X \hookrightarrow \Lambda$ coincides with the \mathcal{O}_X -module structure of E , i.e., it is an \mathcal{O}_X -module E endowed with an action

$$\varphi : \Lambda \otimes_{\mathcal{O}_X} E \longrightarrow E$$

Simpson proved that for every Λ satisfying the previous properties there exists a quasi-projective moduli space of semistable Λ -modules for a certain natural semi-stability condition. Some important examples of moduli spaces that can be constructed as instances of this general theorem include the moduli spaces of vector bundles, Higgs bundles (or, in general, Hitchin pairs/twisted Higgs bundles), connections, logarithmic connections or λ -connections.

On the other hand, let C be a smooth complex projective curve and let D be a finite set of points in C that we will consider as punctures on a Riemann surface. We are interested in studying variants of the previous geometric contraptions over C where we allow the existence of logarithmic singularities over the punctures in D , modulated by a “parabolic” structure over D , i.e., a filtration of the fibers of the underlying sheaf at each of the punctures preserved by the action of the Higgs field or connection. The moduli space of parabolic vector bundles over a curve was described by Mehta and Seshadri [MS80]. Maruyama and Yokogawa generalized the concept of parabolic sheaf to arbitrary dimension and proved the existence of a coarse moduli space of parabolic sheaves [MY92]. Later on, Yokogawa built the moduli space of parabolic Higgs bundles [Yok93]. The moduli space of logarithmic connections (without a parabolic structure) has been built by Nitsure [Nit93] and a notion of moduli space of parabolic connections was developed in [IIS06a] in order to study solutions to the Painlevé VI equation on \mathbb{P}^1 . In this chapter, we adapt the approach of Λ -modules introduced by Simpson to the parabolic scenario in order to unify the previous results in a single theoretical framework and build some similar, yet unknown, moduli spaces such as the parabolic Hodge moduli space, parameterizing parabolic λ -connections.

A parabolic Λ -module is a Λ -module (E, φ) together with a filtration of the fiber $E|_x$ over the each parabolic point $x \in D$

$$E|_x = E_{x,1} \supsetneq E_{x,2} \supsetneq \cdots \supsetneq E_{x,l_x+1} = 0$$

and a sequence of real weights $0 \leq \alpha_{x,1} < \alpha_{x,2} < \cdots < \alpha_{x,l_x} < 1$ such that the action of Λ preserves the filtration in a certain sense. The stability for Λ -modules is substituted by a notion of stability depending on the system of weights $\alpha = \{\alpha_{x,i}\}$ and the filtration $E_\bullet = \{E_{x,i}\}$. The new definition is a natural generalization of existing ones for parabolic vector bundles, parabolic Higgs bundles and parabolic connections and admits the usual constructions such as the Harder-Narasimhan and Jordan-Hölder filtrations. The main result obtained in this part of the thesis is the following (Theorem 3.4.8)

Theorem 3.0.1. *Let Λ be a sheaf of rings of differential operators on $X = C \times S$ over S such that $\Lambda|_{D \times S}$ is locally free. Then there exist a coarse moduli space parameterizing S -equivalence classes of semistable parabolic Λ -modules over (C, D) and an open subset parameterizing isomorphism classes of stable parabolic Λ -modules.*

The first part of the chapter is devoted to reviewing the notion of sheaf of rings of differential operators and Λ -modules as introduced by Simpson [Sim94, §2] and generalizing its properties to the parabolic scenario. Parabolic Λ -modules are defined and we give a notion of stability for parabolic Λ -modules both for complex schemes X of the form $X = C \times S$, over S , where C is a complex projective curve. Versions of the Harder-Narasimhan and Jordan-Hölder filtrations for parabolic Λ -modules are constructed.

The main question treated in section 3.2 is the boundedness of the family of semistable parabolic Λ -modules. We provide uniform bounds for the Mumford-Castelnuovo regularity of both semistable parabolic Λ -modules and destabilizing subsheaves of (possibly unstable) parabolic Λ -modules. We prove several techni-

cal lemmata introducing inequalities over the sections of twists of subsheaves of parabolic Λ -modules.

Section 3.3 describes the construction of a parameterizing space R^{ss} for the family of semistable parabolic Λ -modules. First, we describe a projective scheme parameterizing parabolic quotients of a given sheaf such that the filtrations have a given fixed type. Then, starting from Simpson's rigidification of Λ -modules as quotients of $\Lambda_r \otimes \mathcal{O}_X(-N) \otimes_{\mathbb{C}} \mathbb{C}^{P(N)}$ for a suitable N , we use this “filtered quot scheme” to incorporate the filtration to the parameter space. Finally we prove that the space is a quasi-projective variety that can be embedded into a product of Grassmannians over S using Grothendieck's embedding of the Quot scheme [Gro61].

In section 3.4, we use Geometric Invariant Theory to construct a universal categorical quotient of the previous parameterizing space which corepresents the moduli functor of families of semistable parabolic Λ -modules over X . GIT-semi-stability conditions are computed for the natural action of $\mathrm{SL}(V)$, where V is a complex vector space V , on the product of Grassmannians of the form $\mathrm{Grass}(V \otimes W, p)$ for some vector space W . We use this numerical criterion to describe GIT-semistable parabolic points of R^{ss} and we prove that GIT-semi-stability coincides with slope-stability over the parameter space.

When dealing with parabolic Higgs bundles or parabolic connections, we have a natural notion of residue of the Higgs field or the logarithmic connection at each parabolic point $x \in D$ as the “ -1 coefficient” of the Laurent expansion of the field near the point. In both cases, the residue must preserve the parabolic filtration. Moreover, when we study the geometry of the moduli space of parabolic vector bundles a condition over the residue of the Higgs field or the connection respectively arises naturally. In the case of parabolic Higgs bundles, we usually prescribe the fields to be “strongly parabolic”, so all the eigenvalues are zero. In the case of parabolic connections, if we want them to correspond to “strongly parabolic” Higgs bundles through Simpson's correspondence [Sim90] then the eigenvalues of the residue of the connection must be required to be equal to the corresponding parabolic weight. As Λ -modules are a generalization of these concepts, in section 3.5 we aim to generalize these kinds of requisites to other classes of Λ -modules.

We define the concept of “total residue” of a parabolic Λ -module $(E, E_{\bullet}, \varphi)$ as the morphism

$$\mathrm{Res}(\varphi, x) : \Lambda|_{\{x\} \times S} \otimes_{\mathcal{O}_S} E|_{\{x\} \times S} \longrightarrow E|_{\{x\} \times S}$$

induced by $\varphi : \Lambda \otimes E \rightarrow E$ at the parabolic points. Our definition of parabolic Λ -modules ensures that this map is well defined and preserves the parabolic filtration in the sense that

$$\mathrm{Res}(\varphi, x) (\Lambda|_{\{x\} \times S} \otimes_{\mathcal{O}_S} E_{x,i}) \subseteq E_{x,i}$$

Then, for every section $R \in H^0(S, \Lambda|_{\{x\} \times S})$, the “total residue” induces an endomorphism of the fiber $\mathrm{Res}_R(\varphi, x) \in \mathrm{End}(E|_{\{x\} \times S})$. We prove that the usual notions of residue for parabolic Higgs bundles and parabolic connections can be recovered within this theoretical framework. Then, we define “residual Λ -modules” as the parabolic Λ -modules that satisfying a certain additional condition on the residue analogous to the control of the eigenvalues appearing in parabolic Higgs bundles or parabolic connections. The moduli of “residual Λ -modules” is built as a closed

subscheme of the moduli of parabolic Λ -modules, obtaining the following theorem (Theorem 3.5.3)

Theorem 3.0.2. *There exist a coarse moduli scheme parameterizing S -equivalence classes of semistable “residual” parabolic Λ -modules and an open subset parameterizing isomorphism classes of stable ones.*

In general, the schemes constructed in sections 3.4 and 3.5 are only coarse moduli spaces for the corresponding moduli problems. In section 3.6 we provide a numerical condition which, when satisfied, implies that the subschemes parameterizing stable objects admit a universal family and, therefore, they are fine moduli spaces for their corresponding moduli problems. In particular, we prove the following result (Corollary 3.6.4 of Theorem 3.6.3)

Theorem 3.0.3. *If the system of weights α is full flag, then the moduli spaces of stable parabolic Λ -modules and stable residual parabolic Λ -modules are fine, i.e., they admit a universal family.*

In section 3.7 we apply the previous theorems to the construction of the moduli space of parabolic λ -connections for the group $\mathrm{SL}_r(\mathbb{C})$ (Theorem 3.7.4). We use the deformation to the graduate of the de Rham sheaf of logarithmic differential operators $\Lambda^{\mathrm{DR}, \log D}$ over C with poles over D to obtain a sheaf of differential operators $\Lambda^{\mathrm{DR}, \log D, R}$ over $C \times \mathbb{A}^1$, such that residual parabolic $\Lambda^{\mathrm{DR}, \log D, R}$ -modules over SL correspond to parabolic λ -connections. The fiber over $\lambda = 1$ of $\Lambda^{\mathrm{DR}, \log D, R}$ coincides with $\Lambda^{\mathrm{DR}, \log D}$ and the fiber over $\lambda = 0$ is $\mathrm{Gr}(\Lambda^{\mathrm{DR}, \log D}) \cong \mathrm{Sym}(K^\vee(D)) \cong \Lambda^{\mathrm{Higgs}, \log D}$. We conclude that the constructed moduli space is a quasi projective variety over \mathbb{A}^1 such that its fiber over 0 coincides with the parabolic Higgs moduli space and the fiber over 1 (in fact, over every nonzero λ) is isomorphic to the moduli space of parabolic connections.

Finally, in section 3.8 we analyze the Riemann Hilbert correspondence for the moduli space of parabolic connections. We will prove that, under mild conditions on the parabolic weights, it gives a biholomorphism between the moduli space of parabolic connections on (X, D) and the moduli space of representations of the fundamental group of $X \setminus D$ with certain prescribed monodromies at the parabolic points depending on the choice of the system of weights. This result, together with the construction of the parabolic Hodge moduli space from section 3.7 allows us to construct a parabolic analogue for the Deligne–Hitchin moduli space. The latter result and other further applications and comments about this work are addressed in section 3.9.

3.1 Parabolic Λ -modules

Let $p : X \longrightarrow S$ be any relative smooth projective variety over a complex scheme S .

Definition 3.1.1 (Sheaf of rings of differential operators). *A sheaf of rings of differential operators on X over S is a sheaf of \mathcal{O}_X -algebras Λ over X , with a filtration by sub-algebras $\Lambda_0 \subseteq \Lambda_1 \subseteq \dots$ which satisfies the following properties*

1. $\Lambda = \bigcup_{i=0}^{\infty} \Lambda_i$ and for every i and j , $\Lambda_i \cdot \Lambda_j \subseteq \Lambda_{i+j}$

2. The image of the morphism $\mathcal{O}_X \rightarrow \Lambda$ is equal to Λ_0 .
3. The image of $p^{-1}(\mathcal{O}_S)$ in \mathcal{O}_X is contained in the center of Λ .
4. The left and ring \mathcal{O}_X -module structures on $\mathrm{Gr}_i(\Lambda) := \Lambda_i/\Lambda_{i-1}$ are equal.
5. The sheaves of \mathcal{O}_X -modules $\mathrm{Gr}_i(\Lambda)$ are coherent.
6. The morphism of sheaves

$$\mathrm{Gr}_1(\Lambda) \otimes \cdots \otimes \mathrm{Gr}_1(\Lambda) \rightarrow \mathrm{Gr}_i(\Lambda)$$

induced by the product is surjective.

We will denote by $\Lambda^{\mathrm{DR}} = \mathcal{D}_{X/S}$ the sheaf of differential operators over X relative to S [Ber84]. It represents the main example of sheaf of rings of differential operators and, in fact, the previous set of properties are meant to be an abstraction of the principal characteristics of $\mathcal{D}_{X/S}$. Its graduate $\Lambda^{\mathrm{Higgs}} = \mathrm{Gr}_\bullet(\mathcal{D}_{X/S})$ with the induced sheaf of algebras structure and its deformation to the graduate $\Lambda^{\mathrm{DR},R}$ are additional examples.

Lemma 3.1.2. *Let Λ be a sheaf of rings of differential operators over X . Then for every $i, j \geq 0$*

$$\Lambda_i \cdot \Lambda_j = \Lambda_{i+j}$$

Proof. It is enough to prove that $\underbrace{\Lambda_1 \cdots \Lambda_1}_i = \Lambda_i$, as then

$$\Lambda_i \cdot \Lambda_j = \underbrace{\Lambda_1 \cdots \Lambda_1}_i \cdot \underbrace{\Lambda_1 \cdots \Lambda_1}_j = \underbrace{\Lambda_1 \cdots \Lambda_1}_{i+j} = \Lambda_{i+j}$$

By induction, it is enough to prove that for every i , $\Lambda_i \cdot \Lambda_1 = \Lambda_{i+1}$, i.e., that the morphism $\Lambda_i \otimes \Lambda_1 \rightarrow \Lambda_{i+1}$ is surjective. Let $U \subseteq X$ be open. Let $v \in \Lambda_{i+1}(U)$. As $\underbrace{\mathrm{Gr}_1(\Lambda(U)) \otimes \cdots \otimes \mathrm{Gr}_1(\Lambda(U))}_{i+1} \rightarrow \mathrm{Gr}_{i+1}(\Lambda(U))$ is surjective, there exist

$\overline{w}_{1,1}, \dots, \overline{w}_{l,i+1} \in \mathrm{Gr}_1(\Lambda(U))$ such that

$$\sum_{j=1}^l \overline{w}_{j,1} \cdots \overline{w}_{j,i+1} \cong v \pmod{\Lambda_i(U)}$$

Let $w_{j,i}$ be any representative of $\overline{w}_{j,i}$ in $\Lambda_1(U)$. Then there exists $v' \in \Lambda_i(U)$ such that

$$v = \sum_{j=1}^l w_{j,1} \cdots w_{j,i+1} + v'$$

By induction hypothesis, there exist $v_{1,1}, \dots, v_{m,i} \in \Lambda_1(U)$, such that

$$v' = \sum_{j=1}^m v_{j,1} \cdots v_{j,i}$$

Let $1 \in \Lambda_0(U) \subset \Lambda_1(U)$ be the unity of the ring. Then

$$v = \sum_{j=1}^l w_{j,1} \cdots w_{j,i+1} + \sum_{j=1}^m v_{j,1} \cdots v_{j,i} \cdot 1 \in \underbrace{\Lambda_1(U) \cdots \Lambda_1(U)}_{i+1}$$

□

Definition 3.1.3 (Λ -module). *Let X be an S -scheme. Let Λ be a sheaf of rings of differential operators over X . A Λ -module over X is a sheaf E of left Λ -modules over X such that E is coherent with respect to the structure of \mathcal{O}_X -modules induced by the morphism $\mathcal{O}_x \rightarrow \Lambda_0$.*

Under the previous definition, a vector bundle with an integrable connection can be alternatively described as a locally free Λ^{DR} -module. Similarly, Higgs bundles correspond to locally free Λ^{Higgs} -modules and λ -connections on X correspond to $i_\lambda^* \Lambda^{\text{DR}, R}$ -modules on $X \times \{\lambda\} \subset X \times \mathbb{A}^1$, where $i_\lambda : \{\lambda\} \hookrightarrow \mathbb{A}^1$.

Now, let C be a smooth complex projective curve. Let D be a finite set of points in C . Let S be a complex scheme. Let us consider the complex scheme $X = C \times S$, considered as a relative smooth projective variety over S . Let $\mathcal{O}_X(1) = p^* \mathcal{O}_C(1)$ be an S -very ample invertible sheaf. Let $\bar{D} := D \times S \subset X$. Then it is an effective Cartier divisor on X/S . We are interested in parameterizing certain kinds of geometric objects over X with logarithmic singularities along \bar{D} such as parabolic connections or parabolic Higgs fields. We generalize these notions by enhancing a Λ -module over X with an additional parabolic structure over \bar{D} .

Definition 3.1.4 (Family of parabolic vector bundles). *A family of parabolic vector bundles over (C, D) parameterized by S is a vector bundle E over $C \times S$ together with a weighted flag on $E|_{\{x\} \times S}$ for each $x \in D$ called parabolic structure, i.e., a filtration*

$$E|_{\{x\} \times S} = E_{x,1} \supsetneq E_{x,2} \supsetneq \cdots \supsetneq E_{x,l_x} \supsetneq E_{x,l_x+1} = 0$$

by sub-vector bundles over $\{x\} \times S$ and a system of real weights $0 \leq \alpha_{x,1} < \cdots < \alpha_{x,l_x} < 1$.

We call parabolic type of (E, E_\bullet) to the system of weights $\alpha = \{\alpha_{x,i}\}$ together with the set of ranks $\bar{r} = \{r_{x,i}\}$, $r_{x,i} = \text{rk}(E|_{\{x\} \times S} / E_{x,i})$. A parabolic structure is said to be full flag if $l_x = \text{rk}(E|_x)$ for every parabolic point.

Providing such a filtration on the fibers $E|_{\{x\} \times S}$ is equivalent to giving a weighted filtration of E by subsheaves of the form

$$E = E_x^1 \supsetneq E_x^2 \supsetneq \cdots \supsetneq E_x^{l_x} \supsetneq E_x^{l_x+1} = E(-\{x\} \times S)$$

where for every $x \in D$ and every $i = 1, \dots, l_x$, E_x^i is the sheaf fitting in the following short exact sequence

$$0 \longrightarrow E_x^i \longrightarrow E \longrightarrow E|_{\{x\} \times S} / E_{x,i} \longrightarrow 0$$

Equivalently [MY92, Definition 1.2] we can codify the parabolic structure of a parabolic vector bundle over each parabolic point $x \in D$ as a left continuous real decreasing filtration of sub-sheaves $E_{x,\alpha}$ of E such that

1. For every $x \in D$ and every $\alpha \in \mathbb{R}$, $E_{x,\alpha}$ is coherent and flat over S .
2. $E_{x,0} = E$
3. For every $\alpha \in \mathbb{R}$, $E_{x,\alpha+1} = E_{x,\alpha}(-\{x\} \times S)$

Definition 3.1.5 (Parabolic Λ -module). *Let Λ be a sheaf of rings of differential operators over $X = C \times S$ such that $\Lambda|_{\overline{D}}$ is a locally free $\mathcal{O}_{\overline{D}}$ -module. A parabolic Λ -module over X is a locally free Λ -module E over X flat over S together with a weighted flag on $E|_{\{x\} \times S}$ for each $x \in D$ called parabolic structure, i.e., a filtration*

$$E|_{\{x\} \times S} = E_{x,1} \supsetneq E_{x,2} \supsetneq \cdots \supsetneq E_{x,l_x} \supsetneq E_{x,l_x+1} = 0$$

by sub-vector bundles over $\{x\} \times S$ and a system of real weights $0 \leq \alpha_{x,1} < \cdots < \alpha_{x,l_x} < 1$, such that for every $x \in D$ the filtration $E_{x,i}$ is compatible with the Λ -module structure in the following way. For each $x \in D$ let

$$E = E_x^1 \supsetneq E_x^2 \supsetneq \cdots \supsetneq E_x^{l_x} \supsetneq E_x^{l_x+1} = E(-\{x\} \times S)$$

be the induced filtration of E by subsheaves given by

$$0 \longrightarrow E_x^i \longrightarrow E \longrightarrow E|_{\{x\} \times S} / E_{x,i} \longrightarrow 0$$

Then the image of $\Lambda \otimes E_x^i$ under the morphism $\Lambda \otimes E \longrightarrow E$ lies in E_x^i for every $i = 1, \dots, l_x + 1$.

If $f : T \rightarrow S$ is any S -scheme, a family of parabolic Λ -modules over X parametrized by T , is a parabolic f^Λ -module E over $C \times T$.*

If (E, E_\bullet) is a parabolic Λ -module and $F \subseteq E$ is a vector bundle preserved by Λ , then the parabolic structure E_\bullet induces a structure of parabolic Λ -module on F , taking the filtration

$$F_{x,\bullet} = E_{x,i} \cap F|_{\{x\} \times S}$$

for every $x \in D$. As $E_{x,1} = E|_{\{x\} \times S}$ and $F \subseteq E$, it is clear that $F_\bullet = \{F_{x,i}\}$ defines a parabolic structure on F . Moreover, $E_{x,i}$ and F are preserved by Λ , so F_\bullet is preserved by Λ and (F, F_\bullet) is a parabolic sub- Λ -module.

We will introduce some notation for the basic numerical invariants of a parabolic Λ -module. Let E be a coherent sheaf over X . The Hilbert polynomial of E is $P_E(n) = \chi(E(n))$. By Riemann-Roch theorem, if $X = C \times S$

$$P_E(n) = \deg(E) + \text{rk}(E)(n + 1 - g)$$

Definition 3.1.6. *We define the parabolic degree of a parabolic Λ -module (E, E_\bullet) as the parabolic degree of the underlying parabolic vector bundle*

$$\text{pardeg}(E, E_\bullet) := \deg(E) + \sum_{x \in D} \sum_{i=1}^{l_x} \alpha_{x,i} (\text{rk}(E_{x,i}) - \text{rk}(E_{x,i+1}))$$

We will call the last summand of the previous expression the parabolic weight of (E, E_\bullet) ,

$$\text{wt}(E, E_\bullet) = \text{pardeg}(E, E_\bullet) - \deg(E) = \sum_{x \in D} \sum_{i=1}^{l_x} \alpha_{x,i} (\text{rk}(E_{x,i}) - \text{rk}(E_{x,i+1}))$$

Moreover, we will write for each $x \in D$

$$\text{wt}_x(E, E_\bullet) = \sum_{i=1}^{l_x} \alpha_{x,i} (\text{rk}(E_{x,i}) - \text{rk}(E_{x,i+1}))$$

so $\text{wt}(E, E_\bullet) = \sum_{x \in D} \text{wt}_x(E, E_\bullet)$. In order to simplify the notation, if the parabolic structure is clear from the context, we may write $\text{pardeg}(E)$, $\text{wt}(E)$ and $\text{wt}_x(E)$ to denote the parabolic degree, weight and weight at a point respectively.

Definition 3.1.7. We define the parabolic slope of (E, E_\bullet) as

$$\text{par-}\mu(E) = \frac{\text{pardeg}(E)}{\text{rk}(E)} = \frac{\deg(E) + \text{wt}(E)}{\text{rk}(E)}$$

in order to simplify the notation in subsequent sections, we will write $\eta(E) = \frac{\text{wt}(E)}{\text{rk}(E)}$.

We also define the parabolic Euler characteristic of (E, E_\bullet) as

$$\text{par-}\chi(E) = \chi(E) + \text{wt}(E)$$

The polynomial $\text{par-P}_E(m) := \text{par-}\chi(E(m))$ is called the parabolic Hilbert polynomial of (E, E_\bullet) . Clearly, we can express the polynomial in terms of the Hilbert polynomial of the underlying sheaf E

$$\text{par-P}_E(m) = P_E(m) + \text{wt}(E)$$

Definition 3.1.8 (Slope stability for parabolic Λ -modules). A parabolic Λ -module E over C is said to be (semi-)stable if for every sub- Λ -module F with the induced parabolic structure and $0 < \text{rk}(F) < \text{rk}(E)$

$$\text{par-}\mu(F)(\leq) < \text{par-}\mu(E)$$

Let $p, q \in \mathbb{R}[x]$. By $p(\preceq) \prec q$, we mean that there exists an integer M such that for every $m \geq M$

$$p(m)(\leq) < q(m)$$

Lemma 3.1.9 (Gieseker stability for parabolic Λ -modules). A parabolic Λ -module over C is (semi-)stable if and only if for every sub- Λ -module F with $0 < \text{rk}(F) < \text{rk}(E)$ and the induced parabolic structure

$$\frac{\text{par-P}_F}{\text{rk}(F)}(\preceq) \prec \frac{\text{par-P}_E}{\text{rk}(E)}$$

Proof. By Riemann-Roch Theorem, for every m

$$\text{par-P}_E(m) = P_E(m) + \text{wt}(E) = \chi(E(m)) + \text{wt}(E) = \deg(E) + \text{rk}(E)(m+1-g) + \text{wt}(E)$$

Therefore $\frac{\text{par-P}_F(m)}{\text{rk}(F)} (\leq) < \frac{\text{par-P}_E(m)}{\text{rk}(E)}$ for every big enough m (and, in fact, for any m) if and only if

$$\begin{aligned} \text{par-}\mu(F) + m + 1 - g &= \frac{\deg(F) + \text{wt}(F)}{\text{rk}(F)} + m + 1 - g \\ &(\leq) < \frac{\deg(E) + \text{wt}(E)}{\text{rk}(E)} + m + 1 - g = \text{par-}\mu(E) + m + 1 - g \end{aligned}$$

and this is equivalent to $\text{par-}\mu(F) (\leq) < \text{par-}\mu(E)$. \square

Lemma 3.1.10. *Let (E, E_\bullet) be a parabolic vector bundle and let (F, F_\bullet) be a parabolic subsheaf such that $\text{par-}\mu(E, E_\bullet) = \text{par-}\mu(F, F_\bullet)$. Then F has the induced parabolic structure, i.e., $F_\bullet = E_\bullet \cap F$*

Proof. As $(F, F_\bullet) \subseteq (E, E_\bullet)$, then for every $x \in D$ and $i = 1, \dots, l_x$ we have $F_{x,i} \subseteq E_{x,i} \cap F|_x$. We can rewrite the expression of the parabolic weight of F as

$$\begin{aligned} \text{wt}_x(F, F_\bullet) &= \sum_{i=1}^{l_x} \alpha_{x,i} (\dim(F_{x,i}) - \dim(F_{x,i+1})) \\ &= \sum_{i=2}^{l_x} \dim(F_{x,i}) (\alpha_{x,i} - \alpha_{x,i-1}) + \alpha_1 \dim(F_{x,1}) = \sum_{i=2}^{l_x} \dim(F_{x,i}) (\alpha_{x,i} - \alpha_{x,i-1}) + \alpha_1 \dim(F|_x) \\ &\leq \sum_{i=2}^{l_x} \dim(F|_x \cap E_{x,i}) (\alpha_{x,i} - \alpha_{x,i-1}) + \alpha_1 \dim(F|_x) = \text{wt}_x(F, E_\bullet \cap F) \end{aligned}$$

Therefore

$$\text{par-}\mu(F, F_\bullet) = \mu(F) + \sum_{x \in D} \text{wt}_x(F, F_\bullet) \leq \mu(F) + \sum_{x \in D} \text{wt}_x(F, E_\bullet \cap F) = \text{par-}\mu(F, E_\bullet \cap F)$$

As the parabolic weights are strictly increasing, $\alpha_{x,i} - \alpha_{x,i-1} > 0$, the previous inequalities only become equalities when $\dim(F_{x,i}) = \dim(E_{x,i} \cap F|_x)$ for all $x \in D$ and all $i = 1, \dots, l_x$. \square

Lemma 3.1.11. *Let (E, E_\bullet) be a parabolic sheaf and let T be a torsion subsheaf of E . Let $(\overline{E}, \overline{E}_\bullet)$ be the sheaf $\overline{E} = E/T$ with the induced parabolic structure $\overline{E}_{x,i} = E_{x,i}/(E_{x,i} \cap T|_x)$. Then*

$$\text{par-}\mu(E, E_\bullet) \geq \text{par-}\mu(\overline{E}, \overline{E}_\bullet)$$

and for every $m \in \mathbb{Z}$

$$\frac{h^0(C, E(m)) + \text{wt}(E)}{\text{rk}(E)} \geq \frac{h^0(C, \overline{E}(m)) + \text{wt}(\overline{E})}{\text{rk}(\overline{E})}$$

Proof. We have a short exact sequence

$$0 \longrightarrow T \longrightarrow E \longrightarrow \bar{E} \longrightarrow 0$$

so

$$\deg(\bar{E}) = \deg(E) - \deg(T)$$

On the other hand, as torsion sheaves on a curve are supported in dimension 0

$$\deg(T) = h^0(C, T) - h^1(C, T) = h^0(C, T)$$

So $\deg(\bar{E}) = \deg(E) - h^0(C, T)$. Moreover, as T is torsion, $\text{rk}(\bar{E}) = \text{rk}(E)$.

Now let us consider the parabolic structure. For every $x \in D$ and $i = 1, \dots, l_x$ we have a short exact sequence

$$0 \longrightarrow E_{x,i} \cap T|_x \longrightarrow E_{x,i} \longrightarrow \bar{E}_{x,i} = \frac{E_{x,i}}{E_{x,i} \cap T|_x} = \frac{E_{x,i} + T|_x}{T|_x} \longrightarrow 0$$

Now, taking quotients yields

$$\dim\left(\frac{\bar{E}|_x}{\bar{E}_{x,i}}\right) = \dim\left(\frac{E|_x/T|_x}{(E_{x,i} + T|_x)/T|_x}\right) = \dim\left(\frac{E|_x}{E_{x,i} + T|_x}\right) \geq \dim\left(\frac{E|_x}{E_{x,i}}\right) - \dim T|_x$$

As E and \bar{E} have the same rank yields

$$\dim(\bar{E}_{x,i}) \leq \dim(E_{x,i}) + \dim(T|_x) = \dim(E_{x,i}) + h^0(x, T|_x)$$

Substituting in the weight formula we obtain

$$\begin{aligned} \text{wt}_x(\bar{E}) &= \sum_{i=2}^{l_x} \dim(\bar{E}_{x,i})(\alpha_{x,i} - \alpha_{x,i-1}) + \alpha_{x,1} \dim(\bar{E}_{x,1}) \\ &\leq \sum_{i=2}^{l_x} \dim(E_{x,i})(\alpha_{x,i} - \alpha_{x,i-1}) + \alpha_{x,1} \dim(E_{x,1}) + h^0(x, T|_x) \left(\alpha_{x,1} + \sum_{i=2}^{l_x} (\alpha_{x,i} - \alpha_{x,i-1}) \right) \\ &= \text{wt}_x(E) + h^0(x, T|_x) \alpha_{x,l_x} \leq \text{wt}_x(E) + h^0(x, T|_x) \end{aligned}$$

and equality is only obtained if $h^0(x, T|_x) = 0$. Adding up and taking into account that $h^0(C, T) \geq \sum_{x \in D} h^0(x, T|_x)$ yields

$$\begin{aligned} \text{par-}\mu(\bar{E}) &= \frac{\deg(\bar{E}) + \sum_{x \in D} \text{wt}_x(\bar{E})}{\text{rk}(\bar{E})} = \frac{\deg(E) - h^0(C, T) + \sum_{x \in D} \text{wt}_x(\bar{E})}{\text{rk}(E)} \\ &\leq \frac{\deg(E) + \sum_{x \in D} (\text{wt}_x(\bar{E}) - h^0(x, T|_x))}{\text{rk}(E)} \leq \frac{\deg(E) + \sum_{x \in D} \text{wt}_x(E)}{\text{rk}(E)} = \text{par-}\mu(E) \end{aligned}$$

With regards to the second part of the lemma, from the short exact sequence

$$0 \longrightarrow T(m) \longrightarrow E(m) \longrightarrow \bar{E}(m) \longrightarrow 0$$

we obtain a long exact sequence

$$0 \longrightarrow H^0(C, T(m)) \longrightarrow H^0(C, E(m)) \longrightarrow H^0(C, \bar{E}(m)) \longrightarrow H^1(C, T(m))$$

As T is supported in dimension 0, we have $H^1(C, T(m)) = 0$ and $h^0(C, T(m)) = h^0(C, T)$, so

$$h^0(C, \bar{E}(m)) = h^0(C, E(m)) - h^0(C, T(m)) = h^0(C, E(m)) - h^0(C, T)$$

Now we can repeat the previous argument and we obtain the desired inequality. \square

Corollary 3.1.12. *Let (E, E_\bullet) be a parabolic sheaf and let $(F, F_\bullet) \subseteq (E, E_\bullet)$ be a parabolic subsheaf. Let $(F^{\text{sat}}, F_\bullet^{\text{sat}})$ be the saturation of F in E with the induced parabolic structure from (E, E_\bullet) . Then*

$$\text{par-}\mu(F, F_\bullet) \leq \text{par-}\mu(F^{\text{sat}}, F_\bullet^{\text{sat}})$$

If moreover if for some $m \in \mathbb{Z}$ we have $h^1(C, F(m)) = 0$ then

$$\frac{h^0(C, F(m)) + \text{wt}(F)}{\text{rk}(F)} \leq \frac{h^0(C, F^{\text{sat}}(m)) + \text{wt}(F^{\text{sat}})}{\text{rk}(F^{\text{sat}})}$$

Proof. By Lemma 3.1.10 we may assume without loss of generality that F_\bullet is the induced parabolic structure. For the first part of the corollary, let (Q, Q_\bullet) be the sheaf E/F with the induced parabolic structure. Let (T, T_\bullet) be its torsion with the induced parabolic structure and let $(\bar{Q}, \bar{Q}_\bullet)$ be the torsion free sheaf $Q = (E/F)/T$ with the induced quotient parabolic structure. Then we have the following commutative diagram of parabolic sheaves

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & & (T, T_\bullet) & & \\
 & & & & \downarrow & & \\
 0 & \longrightarrow & (F, F_\bullet) & \longrightarrow & (E, E_\bullet) & \longrightarrow & (Q, Q_\bullet) \longrightarrow 0 \\
 & & \downarrow & & \parallel & & \downarrow \\
 0 & \longrightarrow & (\bar{F}, \bar{F}_\bullet) & \longrightarrow & (E, E_\bullet) & \longrightarrow & (\bar{Q}, \bar{Q}_\bullet) \longrightarrow 0 \\
 & & \downarrow & & & & \downarrow \\
 & & (T, T_\bullet) & & & & 0 \\
 & & \downarrow & & & & \\
 & & 0 & & & &
 \end{array}$$

Where the two rows and columns are exact, so we have

$$\begin{aligned}
 \text{pardeg}(F) &= \text{pardeg}(E) - \text{pardeg}(Q) \\
 \text{pardeg}(\bar{F}) &= \text{pardeg}(E) - \text{pardeg}(\bar{Q})
 \end{aligned}$$

On the other hand, by the previous lemma, we know that $\text{pardeg}(Q) \geq \text{pardeg}(\bar{Q})$. Substituting yields

$$\text{pardeg}(\bar{F}) = \text{pardeg}(F) + \text{pardeg}(Q) - \text{pardeg}(\bar{Q}) \geq \text{pardeg}(F)$$

As $\text{rk}(F) = \text{rk}(\bar{F})$ we obtain $\text{par-}\mu(F) \leq \text{par-}\mu(\bar{F})$.

For the second part of the lemma, observe that from the short exact sequence

$$0 \longrightarrow F(m) \longrightarrow \bar{F}(m) \longrightarrow T(m) \longrightarrow 0$$

we obtain the long exact sequence

$$0 \longrightarrow H^0(C, F(m)) \longrightarrow H^0(C, \bar{F}(m)) \longrightarrow H^0(C, T(m)) \longrightarrow H^1(C, F(m)) = 0$$

Therefore

$$h^0(C, \bar{F}(m)) = h^0(C, F(m)) + h^0(C, T(m)) = h^0(C, F(m)) + h^0(C, T)$$

On the other hand, consider the following commutative diagram of sheaves, where the rows and columns are exact

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & F_x^i & \longrightarrow & \bar{F}_x^i & \longrightarrow & T_x^i \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & F & \longrightarrow & \bar{F} & \longrightarrow & T \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & F|_x/F_{x,i} & & \bar{F}|_x/\bar{F}_{x,i} & & T|_x/T_{x,i} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

Then by the snake lemma we obtain

$$0 \longrightarrow F|_x/F_{x,i} \longrightarrow \bar{F}|_x/\bar{F}_{x,i} \longrightarrow T|_x/T_{x,i} \longrightarrow 0$$

As $\dim(\bar{F}|_x) = \dim(F|_x)$, yields

$$\dim(\bar{F}_{x,i}) = \dim(F_{x,i}) - \dim(T|_x) + \dim(T_{x,i}) \geq \dim(F_{x,i}) - h^0(x, T|_x)$$

Now we can proceed as in the second part of the previous Lemma and the desired inequality follows. \square

We provide some insight on the structure of the subsheaves of a parabolic Λ -module. First of all, the following Lemma allows us to construct saturated parabolic subsheaves of a parabolic Λ -module which are preserved by Λ from any subsheaf.

Lemma 3.1.13. *Let (E, E_\bullet) be a parabolic Λ -module of rank r on X . Suppose that $F \subset E$ is a subsheaf. Then the subbundle $\text{Im}(\Lambda_r \otimes F \rightarrow E)^{\text{sat}}$ with the induced parabolic structure is a parabolic sub- Λ -module.*

Proof. By [Sim94, Lemma 3.2], $G := \text{Im}(\Lambda_r \otimes F \rightarrow E)^{\text{sat}}$ is a subbundle of E preserved by Λ . As (E, E_\bullet) is a parabolic Λ -module, for every parabolic point $x \in D$, the filtration $E_{x \times S, i}$ is preserved by Λ . As G is preserved by Λ , the induced filtration $G_{x, i} = G|_{x \times S} \cap E_{x, i}$ is preserved by Λ , so (G, G_\bullet) is a parabolic sub- Λ -module of (E, E_\bullet) . \square

Theorem 3.1.14 (Harder-Narasimhan filtration). *Suppose that (E, E_\bullet) is a parabolic Λ -module on C . There is a unique filtration by parabolic sub- Λ -modules called the Harder-Narasimhan filtration*

$$0 = (E_0, E_{0,\bullet}) \subsetneq (E_1, E_{1,\bullet}) \subsetneq \dots \subsetneq (E_l, E_{l,\bullet}) = (E, E_\bullet)$$

such that the parabolic quotients $(E_i/E_{i-1}, E_{i,\bullet}/E_{i-1,\bullet})$ are semistable Λ -modules with strictly decreasing parabolic slopes.

Proof. The set of possible slopes of a subsheaf of E is bounded from above. As the set of possible values of the weight of a parabolic sub-sheaf is finite, the set of possible parabolic slopes of parabolic sub- Λ -modules (F, F_\bullet) is bounded from above. Let $\text{par-}\mu_{\max}^\Lambda(E)$ be the maximum parabolic slope of a sub- Λ -module of (E, E_\bullet) . Let (F, F_\bullet) be a sub- Λ -module such that $\text{par-}\mu(F) = \text{par-}\mu_{\max}^\Lambda(E)$. Repeating the argument in Lemma 3.1.10 yields that as F attains the maximum parabolic slope then F must have the induced parabolic structure. Moreover, its saturation $(F^{\text{sat}}, F_\bullet^{\text{sat}})$ is preserved by Λ and has a greater parabolic slope, so F must be saturated. The rank of F is bounded, so we can choose a saturated parabolic sub- Λ -module $(E_1, E_{1,\bullet})$ with $\text{par-}\mu(E_1) = \text{par-}\mu_{\max}^\Lambda(E)$ and maximum rank among those satisfying that condition.

We take $(E_1, E_{1,\bullet})$ as the first step of the Harder-Narasimhan filtration and build the rest of it inductively by applying the previous method to $(E/E_1, E_\bullet/E_{1,\bullet})$. \square

A completely analogous proof to the previous one gives us the following theorem.

Theorem 3.1.15 (Jordan-Hölder filtration). *Let (E, E_\bullet) be a semistable parabolic Λ -module on C over \mathbb{C} . There is a unique filtration by sub- Λ -modules called the Jordan-Hölder filtration*

$$0 = (E_0, E_{0,\bullet}) \subsetneq (E_1, E_{1,\bullet}) \subsetneq \dots \subsetneq (E_l, E_{l,\bullet}) = (E, E_\bullet)$$

such that the parabolic quotients $(E_i/E_{i-1}, E_{i,\bullet}/E_{i-1,\bullet})$ are stable Λ -modules with strictly decreasing parabolic slopes.

We say that two semistable parabolic Λ -modules (E, E_\bullet) and (E', E'_\bullet) are S-equivalent if $\text{Gr}(E, E_\bullet) \cong \text{Gr}(E', E'_\bullet)$, i.e., if they have isomorphic Jordan-Hölder filtrations.

Let S be a complex scheme and let T be a scheme over S . We denote by $X_T = X \times_S T$ the base change of X to T and by Λ_T the base change of Λ to T . By [Sim94, Lemma 2.6] it is a sheaf of rings of differential operators on X_T . In particular, if $\text{Spec}(\mathbb{C}) \cong s \rightarrow S$ is any geometric point, we denote by X_s the fiber of X over s and by Λ_s the base change of Λ to s , which it is a sheaf of rings of differential operators on X_s .

Definition 3.1.16. *A parabolic Λ -module (E, E_\bullet) on $X = C \times S$ is (semi-)stable if the restrictions $(E|_{X_s}, E_\bullet|_{X_s})$ to the geometric fibers X_s are (semi-)stable parabolic Λ_s -modules for every geometric point s of S , all of them with the same Hilbert polynomial and parabolic type.*

Finally, we recall the following notion of “almost” stability due to Maruyama [Mar81]

Definition 3.1.17. A coherent sheaf E on X is said to be of type b , for some $b \in \mathbb{R}$ if for every subsheaf $F \subsetneq E$,

$$\mu(F) \leq \mu(E) + b$$

3.2 Boundedness theorems

The main result proven in this section is the boundedness of the family of semistable parabolic Λ -modules over $C \times S$ with fixed Hilbert polynomial and parabolic type. In order to do so, we prove that every semistable parabolic Λ -module is of type b for a certain uniform b . Then we use Simpson's theorems on Mumford-Castelnuovo regularity for bounded families of sheaves to provide uniform bounds for the regularity of semistable parabolic Λ -modules. Additionally, we find numerical bounds for the number of sections of twists of subsheaves of semistable parabolic Λ -modules. Finally, we obtain some sharper inequalities for the Hilbert polynomial of certain subsheaves of a semistable parabolic Λ -module.

Before introducing the main boundedness theorem, we shall prove two previous technical lemmata.

Lemma 3.2.1. Let $(E_1, E_{1,\bullet})$ and $(E_2, E_{2,\bullet})$ be parabolic vector bundles over X . For every parabolic vector bundle (E, E_\bullet) , let $\text{par-}\mu_{\min}(E, E_\bullet)$ denote the minimum parabolic slope of a parabolic quotient of E . Then

$$\text{par-}\mu_{\min}(E_1 \oplus E_2, E_{1,\bullet} \oplus E_{2,\bullet}) = \min(\text{par-}\mu_{\min}(E_1, E_{1,\bullet}), \text{par-}\mu_{\min}(E_2, E_{2,\bullet}))$$

Proof. Let π_i be the canonical projection of $E_1 \oplus E_2$ to E_i . As every quotient of E_i is a quotient of $E_1 \oplus E_2$, if $(F_i, F_{i,\bullet})$ is a parabolic quotient of $(E_i, E_{i,\bullet})$ such that $\text{par-}\mu(F_i) = \text{par-}\mu_{\min}(E_i)$, then

$$\text{par-}\mu_{\min}(E_1 \oplus E_2, E_{1,\bullet} \oplus E_{2,\bullet}) \leq \text{par-}\mu(F_i, F_{i,\bullet}) = \text{par-}\mu_{\min}(E_i, E_{i,\bullet})$$

Therefore

$$\text{par-}\mu_{\min}(E_1 \oplus E_2, E_{1,\bullet} \oplus E_{2,\bullet}) \leq \min(\text{par-}\mu_{\min}(E_1, E_{1,\bullet}), \text{par-}\mu_{\min}(E_2, E_{2,\bullet}))$$

Let us prove that the opposite inequality holds. Let $f : E_1 \oplus E_2 \twoheadrightarrow F$ be a parabolic quotient such that $\text{par-}\mu(F) = \text{par-}\mu_{\min}(E_1 \oplus E_2)$. By Lemma 3.1.10 F has the induced parabolic structure $F_\bullet = f(E_{1,\bullet} \oplus E_{2,\bullet})$. Consider the following exact commutative diagram of parabolic sheaves with the induced parabolic structures.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_1 & \xrightarrow{i_1} & E_1 \oplus E_2 & \xrightarrow{\pi_2} & E_2 & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow f & & & & \\ 0 & \longrightarrow & f(E_1) & \xrightarrow{i} & F & \longrightarrow & F/f(E_1) & \longrightarrow & 0 \end{array}$$

We have

$$F/f(E_1) = \frac{f(E_1 \oplus E_2)}{f(E_1)} = \frac{f(E_1) + f(E_2)}{f(E_1)} \cong \frac{f(E_2)}{f(E_1) \cap f(E_2)}$$

and for every $x \in D$ and every $i = 1, \dots, l_x$

$$\frac{F_{x,i}}{f(E_1|_x) \cap F_{x,i}} = \frac{f(E_{1,x,i} \oplus E_{2,x,i})}{f(E_1|_x) \cap F_{x,i}} = \frac{f(E_{1,x,i}) + f(E_{2,x,i})}{f(E_1|_x) \cap F_{x,i}} = \frac{f(E_{2,x,i})}{f(E_1|_x) \cap f(E_{2,x,i})}$$

Therefore, $(F/f(E_1), F_\bullet/f(E_1))$ is a parabolic quotient of $(E_2, E_{2,\bullet})$. On the other hand for every $x \in D$ and $i = 1, \dots, l_x$

$$F_{x,i} \cap f(E_1)|_x = (f(E_{1,x,i}) + f(E_{2,x,i})) \cap f(E_1)|_x = f(E_{1,x,i}) + f(E_{2,x,i}) \cap f(E_1)|_x \supseteq f(E_{1,x,i})$$

so $f(E_1)$ with the induced parabolic structure by (F, F_\bullet) is a parabolic quotient of $(E_1, E_{1,\bullet})$. Finally, the second row is exact, so

$$\begin{aligned} \text{par-}\mu(F, F_\bullet) &\geq \min(\text{par-}\mu(f(E_1), F_\bullet \cap f(E_1)), \text{par-}\mu(F/f(E_1), F_\bullet/f(E_1))) \\ &\geq \min(\text{par-}\mu_{\min}(E_1, E_{1,\bullet}), \text{par-}\mu_{\min}(E_2, E_{2,\bullet})) \end{aligned}$$

□

Corollary 3.2.2. *If E is a parabolic vector bundle over X then for every finite dimensional complex vector space V*

$$\text{par-}\mu_{\min}(E) = \text{par-}\mu_{\min}(V \otimes_{\mathbb{C}} E)$$

Proof. Inductively apply the previous lemma taking $E_1 = E^{\oplus n}$ and $E_2 = E$ for $1 \leq n \leq \dim V - 1$. □

Lemma 3.2.3. *Let (E, E_\bullet) be a semistable parabolic Λ -module, and let (F, F_\bullet) be a parabolic sub-bundle of (E, E_\bullet) with the induced parabolic structure. Let $(G_i, G_{i,\bullet})$ denote the image of the morphism of parabolic sheaves $\Lambda_i \otimes F \rightarrow E$. Observe that as $F_\bullet = E_\bullet \cap F$, then*

$$G_{i,\bullet} = \Lambda_i \cdot F_\bullet = \Lambda_i \cdot (E_\bullet \cap F) = (\Lambda_i \cdot E_\bullet) \cap G_i = E_\bullet \cap G_i$$

so G has the induced parabolic structure. For $i = 1, \dots, r$, consider the quotient parabolic sheaf $Q_i = G_i/G_{i-1}$ with the induced parabolic structure. Then for $i = 1, \dots, r$ there exists a surjective morphism of parabolic sheaves

$$\varphi_i : \text{Gr}_1(\Lambda) \otimes_{\mathcal{O}_X} (Q_i, Q_{i,\bullet}) \rightarrow (Q_{i+1}, Q_{i+1,\bullet})$$

and a surjective morphism of parabolic sheaves

$$\varphi_0 : \text{Gr}_1(\Lambda) \otimes_{\mathcal{O}_X} (F, F_\bullet) \rightarrow (Q_1, Q_{1,\bullet})$$

Proof. By Lemma 3.1.2, $\Lambda_1 \cdot \Lambda_i = \Lambda_{i+1}$ for all i , so

$$\Lambda_1 \cdot G_i = \Lambda_1 \cdot \Lambda_i \cdot F = \Lambda_{i+1} \cdot F = G_{i+1}$$

As the previous equation also holds for the corresponding parabolic filtrations, we obtain a surjective morphism of parabolic sheaves

$$\Lambda_1 \otimes (G_i, G_{i,\bullet}) \rightarrow (G_{i+1}, G_{i+1,\bullet}) \rightarrow (Q_{i+1}, Q_{i+1,\bullet})$$

To prove the lemma it is enough to show that the previous morphism factors through the corresponding quotients. Let $U \subseteq X$ be any open subset. Let $\lambda_1, \lambda_2 \in \Lambda_1(U)$ such that $\lambda_2 - \lambda_1 = x$, for some $x \in \Lambda_0(U)$. Let $v_1, v_2 \in G_i(U)$ such that $v_2 - v_1 = w$ for some $w \in G_{i-1}(U)$. Then yields

$$\varphi_i([\lambda_2], [v_2]) = [\lambda_2 \cdot v_2] = [(\lambda_1 + x) \cdot (v_1 + w)] = [\lambda_1 \cdot v_1] + [x \cdot v_2 + \lambda_1 \cdot w]$$

By hypothesis

$$x \cdot v_2 \in \Lambda_0(U) \cdot G_i(U) = \Lambda_0(U) \cdot \Lambda_i(U) \cdot F(U) = \Lambda_i(U) \cdot F(U) = G_i(U)$$

$$\lambda_1 \cdot w \in \Lambda_1(U) \cdot G_{i-1}(U) = G_i(U)$$

Therefore, $x \cdot v_2 + \lambda_1 \cdot w \cong 0 \pmod{G_i(U)}$ and we get that $\varphi_i([\lambda_2], [v_2]) = \varphi_i([\lambda_1], [v_1])$. The given argument also holds for any of the steps of the parabolic filtration, so we get a morphism of parabolic sheaves. The proof of the second part is immediate from the previous computation taking into account that $G_0 = \Lambda_0 \cdot F = F$. \square

Lemma 3.2.4. *The set of semistable parabolic Λ -modules over C with a fixed Hilbert polynomial P and fixed parabolic type is bounded.*

Proof. Let (E, E_\bullet) be a semistable parabolic Λ -module, and let (F, F_\bullet) be a parabolic subsheaf of maximum parabolic slope. By Lemma 3.1.10, F_\bullet is the induced parabolic structure. Let r be the rank of E . As in the previous Lemma, let $(G_i, G_{i,\bullet})$ be the image of $\Lambda_i \otimes F \rightarrow E$, for $i = 1, \dots, r$. Let us denote by (G, G_\bullet) the saturation of $(G_r, G_{r,\bullet})$. By Lemma 3.1.13, (G, G_\bullet) is a parabolic sub- Λ -module, so

$$\text{par-}\mu(G_r) \leq \text{par-}\mu(G) \leq \text{par-}\mu(E)$$

By the previous Lemma, for every $i = 1, \dots, r-1$ there exists a surjection of parabolic sheaves

$$\text{Gr}_1(\Lambda) \otimes_{\mathcal{O}_X} (Q_i, Q_{i,\bullet}) \twoheadrightarrow (Q_{i+1}, Q_{i+1,\bullet})$$

By Serre's vanishing Lemma, there exists an $m \in \mathbb{Z}$ such that $\text{Gr}_1(\Lambda) \otimes_{\mathcal{O}_X}(m)$ is generated by global sections. Let $V = H^0(\text{Gr}_1(\Lambda) \otimes_{\mathcal{O}_X}(m))$. Then we have a surjection

$$V \otimes_{\mathbb{C}} \mathcal{O}_X(-m) \otimes_{\mathcal{O}_X} (G_i/G_{i-1}) \twoheadrightarrow (G_{i+1}/G_i) \quad (3.2.1)$$

Let $(R_i, R_{i,\bullet})$ be a parabolic quotient of $(G_i, G_{i,\bullet})$ such that $\text{par-}\mu(R_i) = \text{par-}\mu_{\min}(G_i)$. Then it has the induced parabolic structure and $(R_i, R_{i,\bullet})$ is a semistable parabolic sheaf, because a destabilizing sheaf for $(R_i, R_{i,\bullet})$ would lead to a parabolic quotient $(R_i, R_{i,\bullet}) \twoheadrightarrow (R'_i, R'_{i,\bullet})$ with less parabolic slope. As any parabolic quotient of $(R_i, R_{i,\bullet})$ is a parabolic quotient of $(G_i, G_{i,\bullet})$, $(R'_i, R'_{i,\bullet})$ would be a quotient of $(G_i, G_{i,\bullet})$ with less parabolic slope than $(R_i, R_{i,\bullet})$, contradicting the minimality assumption.

For each $0 \leq i < r$, if $(R_{i+1}, R_{i+1,\bullet})$ has a nontrivial parabolic subsheaf (H, H_\bullet) which is a parabolic quotient of $(G_i, G_{i,\bullet})$, then by semi-stability of $(R_{i+1}, R_{i+1,\bullet})$,

$\text{par-}\mu(H) \leq \text{par-}\mu(R_{i+1}) = \text{par-}\mu_{\min}(G_{i+1})$. On the other hand, as (H, H_{\bullet}) is a parabolic quotient of $(G_i, G_{i,\bullet})$,

$$\text{par-}\mu_{\min}(G_i) \leq \text{par-}\mu(H) \leq \text{par-}\mu_{\min}(G_{i+1})$$

Otherwise, let $H = \text{Im}(G_i \hookrightarrow G_{i+1} \twoheadrightarrow R_{i+1})$ with the induced parabolic structure. It is a parabolic subsheaf of $(R_{i+1}, R_{i+1,\bullet})$ which is a quotient of $(G_i, G_{i,\bullet})$, so $H = 0$. Then, $(R_{i+1}, R_{i+1,\bullet})$ is a parabolic quotient of $(Q_{i+1}, Q_{i+1,\bullet})$. If $i > 0$, surjection (3.2.1) implies that $(R_{i+1}, R_{i+1,\bullet})$ is a parabolic quotient of $V \otimes_{\mathbb{C}} \mathcal{O}_X(-m) \otimes_{\mathcal{O}_X} (Q_i, Q_{i,\bullet})$. Therefore, we get a parabolic quotient

$$V \otimes_{\mathbb{C}} G_i \twoheadrightarrow V \otimes_{\mathbb{C}} (Q_i, Q_{i,\bullet}) \twoheadrightarrow (R_{i+1}(m), R_{i+1,\bullet}(m))$$

By Corollary 3.2.2,

$$\begin{aligned} \text{par-}\mu_{\min}(G_i) &= \text{par-}\mu_{\min}(V \otimes_{\mathbb{C}} G_i) \leq \text{par-}\mu(R_{i+1}(m)) = \\ &\text{par-}\mu(R_{i+1}) + m = \text{par-}\mu_{\min}(G_{i+1}) + m \end{aligned}$$

For $i = 0$, from the Lemma we get a surjection

$$V \otimes_{\mathbb{C}} \mathcal{O}_X(m) \otimes_{\mathcal{O}_X} (F, F_{\bullet}) \twoheadrightarrow (Q_1, Q_{1,\bullet})$$

so by the same argument $\text{par-}\mu_{\min}(F) \leq \text{par-}\mu_{\min}(G_1) + m$. Combining all the previous inequalities for $i = 0, \dots, r-1$, we conclude that $\text{par-}\mu_{\min}(F) \leq \text{par-}\mu_{\min}(G_r) + rm \leq \text{par-}\mu(E) + rm$. As (F, F_{\bullet}) is the parabolic subsheaf with maximum parabolic slope, every parabolic quotient of (F, F_{\bullet}) must have bigger or equal parabolic slope, so $\text{par-}\mu(F) \leq \text{par-}\mu_{\min}(F) \leq \text{par-}\mu(E) + rm$. Therefore, for every parabolic subsheaf $(F', F'_{\bullet}) \subseteq (E, E_{\bullet})$, $\mu(F') + \text{wt}(F') \leq \text{par-}\mu(F) \leq \text{par-}\mu(E) + rm = \mu(E) + \text{wt}(E) + rm$. As $\text{wt}(F') \geq 0$ for all parabolic sheaves, yields

$$\mu(F') \leq \mu(E) + rm + \text{wt}(E) - \text{wt}(F') \leq \mu(E) + rm + \text{wt}(E)$$

Every subsheaf $F' \subseteq E$ can be given the induced parabolic structure, so the previous inequality proves that there exists a number $b \in \mathbb{R}$ such that for every semistable parabolic Λ -module (E, E_{\bullet}) over X flat over S of rank r and the given parabolic type and every subsheaf $F' \subseteq E$, $\mu(F') \leq \mu(E) + b$. By [Mar81, Theorem 2.6], the set of sheaves underlying a semistable parabolic Λ -module with Hilbert polynomial P and the given parabolic structure is bounded. Given one such sheaf, the parabolic Λ -module structure is uniquely determined by a suitable element of $Fl(E|_{\{x\} \times S})$ for each $x \in D$, and a morphism $\text{Hom}(\Lambda_1 \otimes_{\mathcal{O}_X} E, E)$, so the set of semistable parabolic Λ -modules is bounded. \square

We can extend the previous lemma to the relative case.

Lemma 3.2.5. *The set of semistable parabolic Λ -modules over $X = C \times S$ with a fixed Hilbert polynomial P and fixed parabolic type is bounded.*

Proof. By [Sim94, Proposition 3.5], the number $m \in \mathbb{Z}$ fixed in the previous proof can be chosen so that it works uniformly over all geometric points $s \in S$. Therefore, the upper bound on $\mu(E) - \mu(F)$ in the previous Lemma holds over every $s \in S$. Then the boundedness is a consequence of [Sim94, Theorem 1.1]. \square

Given a sheaf F on X flat over S , if $\pi : X \rightarrow S$ then we write

$$H^i(X/S, F) = R^i \pi_* F$$

Corollary 3.2.6. *Let $X = C \times S$. There exists an integer N depending only on X , P and the parabolic weights such that for every S -scheme S' , every $n \geq N$ and every semistable parabolic Λ -module (E, E_\bullet) over $X' := C \times S'$*

1. *For all $i > 0$ $H^i(X'/S', E(n)) = 0$*

2. *The morphism*

$$H^0(X'/S', E(n)) \otimes_{\mathcal{O}_{S'}} \mathcal{O}_{X'}(-n) \rightarrow E$$

is surjective.

3. *$H^p(X'/S', E(n))$ is locally free over S' and commutes with base change, in the sense that if $f : S'' \rightarrow S'$ is an S -morphism, then*

$$f^* H^0(X'/S', E(n)) \cong H^0(C \times S''/S'', f^* E(n))$$

Proof. It holds as a consequence of the previous boundedness Lemma and [Sim94, Lemma 1.9]. \square

Now we will introduce some Lemmas providing bounds on the cohomology of semistable parabolic Λ -modules and its subsheaves.

Lemma 3.2.7. *There exists a number $B \in \mathbb{R}$ depending on e , r and X such that if E is a torsion free sheaf of type e and rank r on a fiber X_s , then for all k*

$$h^0(X_s, E(k)) \leq r [(\mu(E) + k + B)]^+$$

where $[a]^+ = \max(a, 0)$ for $a \in \mathbb{R}$.

Proof. Let $0 = F_0 \subsetneq F_1 \subsetneq \dots \subsetneq F_l = E$ be the Harder-Narasimhan filtration of E as a sheaf over X_s . As E is of type e , then for every $i = 1, \dots, l$,

$$\mu(F_i/F_{i-1}) \leq \mu(E) + e$$

On the other hand, as F_i/F_{i-1} are semistable torsion free sheaves on X_s , by [Sim94, Lemma 1.7], there exists a number B_{r_i} depending only on $r_i := \text{rk}(F_i/F_{i-1})$, such that

$$h^0(X_s, (F_i/F_{i-1})(k)) \leq \text{rk}(F_i/F_{i-1}) [\mu(F_i/F_{i-1}) + k + B_{r_i}]^+$$

As the set of possible ranks for F_i/F_{i-1} is bounded, taking $B' = \max_{k=1, \dots, r} (B_k)$, yields

$$\begin{aligned} h^0(X_s, (F_i/F_{i-1})(k)) &\leq \text{rk}(F_i/F_{i-1}) [\mu(F_i/F_{i-1}) + k + B']^+ \\ &\leq \text{rk}(F_i/F_{i-1}) [\mu(E) + e + k + B']^+ \end{aligned}$$

for every $i = 1, \dots, l$. Therefore

$$h^0(X_s, E(k)) \leq \sum_{i=1}^l h^0(X_s, (F_i/F_{i-1})(k)) \leq \sum_{i=1}^l \text{rk}(F_i/F_{i-1}) [\mu(E) + e + k + B']^+$$

As $\sum_{i=1}^l \text{rk}(F_i/F_{i-1}) = r$, taking $B = e + B'$ we obtain the desired bound. \square

Corollary 3.2.8. *There exists a number $B \in \mathbb{R}$ depending on Λ , r , the parabolic type and X such that if E is a semistable parabolic Λ -module of rank r and the given parabolic type on a geometric fiber X_s , then for all k*

$$h^0(X_s, E(k)) \leq r [(\mu(E) + k + B)]^+$$

where $[a]^+ = \max(a, 0)$ for $a \in \mathbb{R}$.

Proof. By Lemma 3.2.5, the set of coherent sheaves underlying a semistable parabolic Λ -module of the given parabolic type over a geometric fiber X_s is contained in the set of coherent sheaves of type b , for some number b depending only on Λ , r , the parabolic type and X . Then, the results yields as a consequence of the previous Lemma. \square

Now, we provide an extension on Corollary 3.2.6 allowing us find uniform bounds for the Serre vanishing theorem on destabilizing subsheaves of any given parabolic Λ -module.

Lemma 3.2.9. *Let $b, e \in \mathbb{R}$. There exists an integer N depending only on b , e , r and X such that if E is a torsion free sheaf of type e and slope $\mu(E) \geq b$ on a geometric fiber X_s then for every $n \geq N$*

1. $h^1(X_s, E(n)) = 0$
2. $E(n)$ is generated by global sections.

Proof. Let E be a torsion free sheaf of type e and slope $\mu(E) \geq b$ on X_s . Let us prove that $K \otimes E^\vee$ is of type e' for some e' depending only on r and the genus g of X_s , where K is the canonical bundle on X_s . Let F be a subsheaf of $K \otimes E^\vee$ of maximum slope. In particular, F must be saturated. As both sheaves are torsion free, taking duals, $K^\vee \otimes F^\vee$ is a quotient of E . Let G be the kernel of the quotient.

$$\begin{array}{ccccccc} 0 & \longrightarrow & F & \longrightarrow & K \otimes E^\vee & & \\ & & & & & & \\ 0 & \longrightarrow & G & \longrightarrow & E & \longrightarrow & K^\vee \otimes F^\vee \longrightarrow 0 \end{array}$$

E is of type e , so

$$\deg(G) \leq \operatorname{rk}(G)\mu(E) + \operatorname{rk}(G)e$$

On the other hand, by additivity of the degree

$$\deg(K^\vee \otimes F^\vee) = \deg(E) - \deg(G) \geq \deg(E) - \operatorname{rk}(G)\mu(E) - \operatorname{rk}(G)e$$

Dividing by the rank and taking into account that $\operatorname{rk}(K^\vee \otimes F^\vee) = \operatorname{rk}(F) = \operatorname{rk}(E) - \operatorname{rk}(G)$ yields,

$$\mu(K^\vee \otimes F^\vee) \geq \frac{\operatorname{rk}(E)}{\operatorname{rk}(F)}\mu(E) - \frac{\operatorname{rk}(G)}{\operatorname{rk}(F)}\mu(E) - \frac{\operatorname{rk}(G)}{\operatorname{rk}(F)}e = \mu(E) - \frac{\operatorname{rk}(G)}{\operatorname{rk}(F)}e \geq \mu(E) - \operatorname{rk}(E)e$$

Computing the slope of the left hand side results in

$$\mu(E) - \operatorname{rk}(E)e \leq \mu(K^\vee \otimes F^\vee) = \mu(F^\vee) - 2(g-1) = -\mu(F) - 2(g-1)$$

Therefore

$$\begin{aligned}\mu(F) &\leq -\mu(E) + \operatorname{rk}(E)e - 2(g-1) \\ &= \mu(K \otimes E^\vee) + \operatorname{rk}(E)e + 2(g-1) - 2(g-1) = \mu(K \otimes E^\vee) + \operatorname{rk}(E)e\end{aligned}$$

Taking $e' = \operatorname{rk}(E)e$ we get that $K \otimes E^\vee$ is of type e' .

By Lemma 3.2.7 there exists a number B depending only on e' , r and X such that for every n , $h^0(X_s, K \otimes E^\vee(-n)) = 0$ if $0 \geq \mu(K \otimes E^\vee) - k + B = 2\operatorname{rk}(E)(g-1) - \mu(E) - n + B$. As $\mu(E) \geq b$, if we fix an N such that $0 \geq 2\operatorname{rk}(E)(g-1) - b - N + B$, then for every $n \geq N$

$$0 \geq 2\operatorname{rk}(E)(g-1) - b - N + B \geq 2\operatorname{rk}(E)(g-1) - \mu(E) - n + B$$

so for every $n \geq N$, $h^1(X_s, E(n)) = h^0(X_s, K \otimes E^\vee(-n)) = 0$. This yields the desired bound for the first part of the lemma. As the dimension of X_s is 1, (i) is equivalent to E being $(N+1)$ -regular in the sense of Mumford-Castelnuovo. By [HL96, Lemma 1.7.2], $E(n)$ is generated by global sections for every $n \geq N+1$. \square

Corollary 3.2.10. *There exists an integer N depending on Λ , P , the parabolic type and X such that for every $n \geq N$, and every parabolic Λ -module (E, E_\bullet) over a geometric fiber X_s with Hilbert polynomial P and fixed parabolic type, if (F, F_\bullet) is a parabolic sub- Λ -module of (E, E_\bullet) with maximum slope then*

$$1. \ h^1(X_s, F(n)) = 0$$

2. $F(n)$ is generated by global sections. In particular

$$H^0(X_s, F(n)) \otimes \mathcal{O}_{X_s}(-n) \longrightarrow F$$

is surjective.

Proof. Let r denote the rank of any (and therefore all) of the considered parabolic Λ -modules E . Let (F, F_\bullet) a parabolic sub- Λ -module with maximum slope. Then it is semistable as a parabolic Λ -module. In particular, it is a torsion free sheaf of type b for the constant b given by Lemma 3.2.5 such that $\operatorname{par}\text{-}\mu(F) \geq \operatorname{par}\text{-}\mu(E)$. The set of possible values of $\operatorname{wt}(F)$ is bounded, as

$$\operatorname{wt}(F) = \frac{\sum_{x \in D} \sum_{i \in \alpha_{F,x}} \alpha_{x,i}}{\operatorname{rk}(F)} \leq \sum_{x \in D} \sum_{i=1}^{l_x} \alpha_{x,i}$$

where $\alpha_{F,x}$ is the set of indexes $i \in \{1, \dots, l_x\}$ such that $E_{x,i} \cap F|_{\{x\} \times S} \neq E_{x,i+1} \cap F|_{\{x\} \times S}$. Calling wt_{\max} to the right hand side of the inequality, yields

$$\mu(F) \geq \operatorname{par}\text{-}\mu(E) - \operatorname{wt}(F) \geq \operatorname{par}\text{-}\mu(E) - \operatorname{wt}_{\max}$$

By the previous Lemma, there exists an integer $N_{\operatorname{rk}(F)}$ depending only on b , $\operatorname{par}\text{-}\mu(E) - \operatorname{wt}_{\max}$, $\operatorname{rk}(F)$ and X such that for $n \geq N_{\operatorname{rk}(F)}$, $F(n)$ is acyclic and it is generated by global sections. As $0 < \operatorname{rk}(F) \leq r$, it is enough to take $N = \max_{i=1}^r (N_i)$. \square

Now we will provide some sharp inequalities for the Hilbert polynomials of subsheaves of a semistable parabolic Λ -module. Let (E, E_\bullet) be any Λ -module over a geometric fiber X_s and (F, F_\bullet) be a subsheaf. If (E, E_\bullet) is semistable and (F, F_\bullet) is preserved by Λ , semi-stability condition implies that for every n

$$\frac{h^0(X_s, F(n)) - h^1(X_s, F(n))}{\text{rk}(F)} + \eta(F) = \frac{P_F(n)}{\text{rk}(F)} + \eta(F) \leq \frac{P_E(n)}{\text{rk}(E)} + \eta(E)$$

We will prove that for big enough n the previous inequality can be sharpened by removing the term $h^1(X_s, F(n))$. The equality case for the sharpened inequality will be analyzed.

Finally, we will prove an inequality implying that if $E(n)$ is generated by global sections and the previous inequality is strict then there exists a uniform lower bound for the difference between its right hand side and its left hand side.

Lemma 3.2.11. *There exists an integer N such that for every $n \geq N$ if (E, E_\bullet) is a semistable parabolic Λ -module with Hilbert polynomial P and fixed parabolic type, then for every parabolic subsheaf $(F, F_\bullet) \subsetneq (E, E_\bullet)$ such that its saturation is a parabolic sub- Λ -module and every $n \geq N$*

$$\frac{h^0(X_s, F(n))}{\text{rk}(F)} + \eta(F) \leq \frac{P_E(n)}{\text{rk}(E)} + \eta(E)$$

Moreover, if equality holds for some $n \geq N$ then F is saturated and we have $h^1(X_s, F(n)) = 0$.

Proof. Let $(F, F_\bullet) \subsetneq (E, E_\bullet)$ be a parabolic subsheaf. Without loss of generality we can assume that F has the induced parabolic structure. Let $0 = G_0 \subsetneq G_1 \subsetneq \dots \subsetneq G_l = F$ be the Harder-Narasimhan filtration of F as a subsheaf of E . For every $i = 1, \dots, l$, G_i/G_{i-1} is a semistable sheaf, so, as before, there exist integers B_{r_i} depending only on $r_i := \text{rk}(G_i/G_{i-1})$ and X such that for every n and every $i = 1, \dots, l$

$$h^0(X_s, (G_i/G_{i-1})(n)) \leq \text{rk}(G_i/G_{i-1}) [\mu(G_i/G_{i-1}) + n + B_{r_i}]^+$$

Let $B = \min_{k=1, \dots, r} B_k$. Then for every i yields

$$h^0(X_s, F(n)) \leq \sum_{i=1}^l h^0(X_s, (G_i/G_{i-1})(n)) \leq \sum_{i=1}^l r_i [\mu(G_i/G_{i-1}) + n + B]^+ \quad (3.2.2)$$

As (E, E_\bullet) is a semistable parabolic Λ -module, by Lemma 3.2.5 there exists a number b depending only on Λ , r , the parabolic type and X , such that $\mu(G_i/G_{i-1}) \leq \mu(E) + b$. On the other hand, let $\nu(F) = \min_{i=1, \dots, l} (\mu(G_i/G_{i-1})) = \mu(G_1)$. Then, substituting the bounds in equation (3.2.2) and taking into account that $r_1 \geq 1$ and $\sum_{i=1}^l r_i = \text{rk}(F)$ yields

$$h^0(X_s, F(n)) \leq (\text{rk}(F) - 1) [\mu(E) + b + n + B]^+ + [\nu(F) + n + B]^+$$

Now, suppose that $\nu(F) \leq \mu(E) - C$ for some $C \geq 0$. Then for $n \geq C - \mu(E) - B = N_1(C)$ we have

$$\begin{aligned} h^0(X_s, F(n)) &\leq (\text{rk}(F) - 1)(\mu(E) + b + n + B) + \mu(E) - C + n + B = \\ &\quad \text{rk}(F)(n + \mu(E) + B) + (\text{rk}(F) - 1)b - C \end{aligned}$$

Both $\text{rk}(F)$ and $\eta(F)$ are bounded uniformly over each choice of E and F . Therefore, there exists C big enough so that for $n \geq C - \mu(E) - B = N_1(C)$

$$h^0(X_s, F(n)) \leq \text{rk}(F)(n + \mu(E) + B) + (\text{rk}(F) - 1)b - C < \text{rk}(F)(n + 1 - g + \text{par-}\mu(E) - \eta(F))$$

Then for $n \geq N_1(C)$

$$\frac{h^0(X_s, F(n))}{\text{rk}(F)} + \eta(F) < n + 1 - g + \text{par-}\mu(E) = \frac{P_E(n)}{\text{rk}(E)} + \eta(E)$$

Therefore, there exist positive numbers C and $N_1 = N_1(C)$ depending only on Λ , r , the parabolic type, P and X such that the Lemma holds for the given N_1 for every subsheaf F such that $\nu(F) \leq \mu(E) - C$. Let us suppose that (F, F_\bullet) is a parabolic subsheaf whose saturation is a parabolic sub- Λ -module and such that $\nu(F) \geq \mu(E) - C$. As the slope is invariant under Jordan equivalence

$$\mu(F) = \mu\left(\bigoplus_{i=1}^r (G_i/G_{i-1})\right) \geq \nu(F) \geq \mu(E) - C$$

Moreover, for every subsheaf $G \subsetneq F \subsetneq E$

$$\mu(G) \leq \mu(E) + b \leq \mu(F) + C + b$$

Therefore, F is a torsion free sheaf of type $b + C$ with $\mu(F) \geq \mu(E) - C$. By Lemma 3.2.9, there exists an integer N_2 depending on $b + C$, $\mu(E) - C$ and X , i.e., on Λ , P , X and the parabolic type, such that for every $n \geq N_2$, $h^1(X_s, F(n)) = 0$. By Corollary 3.1.12 for every such sheaf F

$$\frac{h^0(X_s, F(n))}{\text{rk}(F)} + \eta(F) \leq \frac{h^0(X_s, F^{\text{sat}}(n))}{\text{rk}(F^{\text{sat}})} + \eta(F^{\text{sat}})$$

so we may assume without loss of generality that F is a saturated sub- Λ -module. As (F, F_\bullet) is a parabolic sub- Λ -module of (E, E_\bullet) , by semi-stability, for $n \geq N_2$

$$\frac{h^0(F(n))}{\text{rk}(F)} + \eta(F) = \frac{P_F(n) - h^1(F(n))}{\text{rk}(F)} + \eta(F) = \frac{P_F(n)}{\text{rk}(F)} + \eta(F) \leq \frac{P_E(n)}{\text{rk}(E)} + \eta(E)$$

Then, it is enough to pick $N = \max(N_1, N_2)$ to get the first part of the result for every parabolic sub- Λ -module. Now suppose that for some $n \geq N$

$$\frac{h^0(X_s, F(n))}{\text{rk}(F)} + \eta(F) = \frac{h^0(X_s, F^{\text{sat}}(n))}{\text{rk}(F^{\text{sat}})} + \eta(F^{\text{sat}}) = \frac{P_E(n)}{\text{rk}(E)} + \eta(E)$$

Then $h^0(X_s, F(n)) = h^0(X_s, F^{\text{sat}}(n))$. By the choice of C through the proof, the only option for equality to hold is that $\mu(F) \geq \mu(E) - C$, so $F^{\text{sat}}(n)$ is generated by global sections. Therefore, $F(n)$ is also generated by global sections and, hence, $F = F^{\text{sat}}$. By the choice of N , we know that

$$h^1(X_s, F(n)) = h^1(X_s, F^{\text{sat}}(n)) = 0$$

□

Lemma 3.2.12. *Let us fix a certain parabolic type. Then there exists a real number $\delta > 0$ such that for every parabolic sheaf (E, E_\bullet) of Hilbert polynomial P of the given parabolic type, if $(F, F_\bullet) \subsetneq (E, E_\bullet)$ is a subsheaf such that*

$$\frac{h^0(X_s, F(n))}{\text{rk}(F)} + \eta(F) < \frac{h^0(X_s, E(n))}{\text{rk}(E)} + \eta(E)$$

then

$$\frac{h^0(X_s, F(n))}{\text{rk}(F)} + \eta(F) + \delta \leq \frac{h^0(X_s, E(n))}{\text{rk}(E)} + \eta(E)$$

Proof. The left and right side of the inequality are sums of rational numbers, so its difference is a positive rational number p/q , with $p, q > 0$ coprime, whose denominator is at most the least common multiple of the denominators of all the summands appearing in the expression. Therefore

$$\begin{aligned} \frac{h^0(X_s, E(n))}{\text{rk}(E)} + \eta(E) - \frac{h^0(X_s, F(n))}{\text{rk}(F)} - \eta(F) \\ \geq \frac{1}{\text{LCM}\{\text{rk}(F)q_{x,i}, \text{rk}(E)q_{x,i}\}} \geq \frac{1}{r! \prod_{x \in D} \prod_{i=1}^{l_x} q_{x,i}} \end{aligned}$$

where $\alpha_{x,i} = \frac{p_{x,i}}{q_{x,i}}$ with $p_{x,i} \in \mathbb{Z}^{\geq 0}$ and $q_{x,i} \in \mathbb{Z}^{> 0}$. \square

3.3 Parameterizing scheme for parabolic Λ -modules

Given a scheme X over S , a coherent sheaf F over X and a polynomial P , let $\text{Quot}_{X/S}(F, P)$ denote the Quot scheme of quotients of F over X flat over S with Hilbert polynomial P . It is a projective scheme representing the moduli functor $\mathfrak{Quot}_{X/S}(F, P) : (\text{Sch}_S) \rightarrow (\text{Sets})$ that assigns each $f : T \rightarrow S$ the set of quotients $f^*F \twoheadrightarrow Q$ over $X \times_S T$ flat over T with Hilbert polynomial P . Let $\mathfrak{Quot}_{X/S}^{\text{LF}}(F, P) : (\text{Sch}_S) \rightarrow (\text{Sets})$ be the subfunctor of families of locally free quotients, and let $\text{Quot}_{X/S}^{\text{LF}}(F, P) \subseteq \text{Quot}_{X/S}(F, P)$ be the open subscheme representing such subfunctor.

Lemma 3.3.1. *Let E and F be coherent sheaves over X flat over S such that there exists a surjective morphism $p : E \twoheadrightarrow F$. Then $p^* : \text{Quot}_{X/S}(F, P) \rightarrow \text{Quot}_{X/S}(E, P)$ is a closed embedding.*

Proof. Let $K = \text{Ker}(p)$. Let $f : T \rightarrow S$ and let $(G, \psi_G) \in \mathfrak{Quot}_{X/S}(E, P)(T)$, where $\psi_G : f^*E \twoheadrightarrow G$. As F is flat over S , $\text{Ker}(f^*E \rightarrow f^*F) = f^*K$. Then ψ_G factors through the pullback of F if and only if the image of f^*K by the quotient $\psi_G : f^*E \twoheadrightarrow G$ is zero.

$$\begin{array}{ccccc} 0 & & & & \\ \downarrow & & & & \\ f^*K & & & & \\ \downarrow & \searrow & & & \\ f^*E & \twoheadrightarrow & G & \rightarrow & 0 \\ \downarrow & \nearrow & & & \\ f^*F & & & & \\ \downarrow & & & & \\ 0 & & & & \end{array}$$

Let (G_E, ψ_E) be the universal quotient of $\text{Quot}_{X/S}(E, P)$ and $g : T \rightarrow \text{Quot}_{X/S}(E, P)$ be the morphism corresponding to (G, ψ_G) . Then (G, ψ_G) belongs to the image of $\text{Quot}_{X/S}(F, P)$ if and only if the pullback of $\pi_X^* K \rightarrow G_E$ by g is zero, where $\pi_X : X \times_S \text{Quot}_{X/S}(E, P) \rightarrow X$ is the projection to the first factor. By [Yok93, Lemma 4.3], there is a closed subscheme Z of $\text{Quot}_{X/S}(E, P)$ such that the pullback is zero if and only if g factors through Z . Therefore, the image of $\text{Quot}_{X/S}(F, P)$ is closed.

Let $f' : T' \rightarrow S$ be another S -scheme and let $\varphi : T' \rightarrow T$ be a morphism of S -schemes. Let us prove that the following diagram is commutative.

$$\begin{array}{ccc} \mathfrak{Quot}_{X/S}(F, P)(T) & \xrightarrow{\varphi^*} & \mathfrak{Quot}_{X/S}(F, P)(T') \\ \downarrow p^* & & \downarrow p^* \\ \mathfrak{Quot}_{X/S}(E, P)(T) & \xrightarrow{\varphi^*} & \mathfrak{Quot}_{X/S}(E, P)(T') \end{array}$$

An element of $\mathfrak{Quot}_{X/S}(F, P)(T)$ is given as a quotient $f^* F \twoheadrightarrow G$ over $X \times_S T$, which is a quotient

$$f^* E \twoheadrightarrow f^* F \twoheadrightarrow G$$

As the pullback is right exact, its image by φ^* is a quotient

$$\begin{array}{ccccc} \varphi^* f^* E & \twoheadrightarrow & \varphi^* f^* F & \twoheadrightarrow & \varphi^* G \\ \parallel & & \parallel & & \\ f'^* E & \twoheadrightarrow & f'^* F & & \end{array}$$

The pullback by φ of the composition $f^* E \rightarrow f^* F \rightarrow G$ is the composition of the pullbacks $\varphi^* f^* E \rightarrow \varphi^* f^* F \rightarrow \varphi^* G$, so we get that $(p^* \circ \varphi^*)(G) = (\varphi^* \circ p^*)(G)$. Therefore, p^* induces a natural transformation $\mathfrak{Quot}_{X/S}(F, P) \rightarrow \mathfrak{Quot}_{X/S}(E, P)$ and p^* is a closed embedding. \square

Given schemes $X_i \rightarrow S$ for $i = 1, \dots, n$, let

$$\prod_S^n X_i$$

denote the fiber product of the X_i over S .

Lemma 3.3.2. *Let P be a fixed Hilbert polynomial, with leading coefficient r , and let $\bar{r} = \{r_{x,i}\}$ for $x \in D$, $1 < i \leq l_x$ be integers. Let F be a coherent sheaf over $X = C \times S$ flat over S such that $F|_{\bar{D}}$ is locally free. Let $\mathfrak{FQuot}_{X/S}^{\text{LF}}(F, P, \bar{r})$ be the functor that associates each S -scheme T the set of isomorphism classes of pairs (E, E_\bullet) consisting on a locally free quotient sheaf E of the pullback of F over $X_T = C \times T$ flat over T with Hilbert polynomial $P_E = P$ and a filtration by subbundles over T*

$$E|_{\{x\} \times T} = E_{x,1} \supsetneq E_{x,2} \supsetneq \dots \supsetneq E_{x,l_x}$$

for each $x \in D$ such that for each $1 < i \leq l_x$, $\text{rk}(E|_{x \times T}/E_{x,i}) = r_{x,i}$. Then there is a closed subscheme $\text{FQuot}_{X/S}^{\text{LF}}(F, P, \bar{r})$ of

$$\text{Quot}_{X/S}^{\text{LF}}(F, P) \times_S \prod_{x \in D} \text{Grass}(F|_{\{x\} \times S}, r) \times_S \prod_{x \in D} \prod_{i=2}^{l_x} \text{Grass}(F|_{\{x\} \times S}, r_{x,i})$$

over $\text{Quot}_{X/S}^{\text{LF}}(F, P)$ representing $\mathfrak{FQuot}_{X/S}^{\text{LF}}(F, P, \bar{r})$.

Proof. Let $p : F \rightarrow E$ be the universal quotient of $Q := \text{Quot}_{X/S}^{\text{LF}}(F, P)$, and let $\pi_Q : Q \rightarrow S$. Suppose that D consists on a single closed point $x \in C$. As E is locally free and $F|_{\{x\} \times S}$ is locally free, $E|_{\{x\} \times Q}$ is a locally free quotient of $\pi_Q^* F|_{\{x\} \times Q}$ of rank r , so it represents a Q -point $e : Q \rightarrow \text{Grass}(F|_{\{x\} \times S}, r)$. Therefore, the graph of e is a closed subscheme \tilde{Q} of $\text{Quot}_{X/S}^{\text{LF}}(F, P) \times_S \text{Grass}(F|_{\{x\} \times S}, r)$ over Q corresponding to the family of pairs of a quotient sheaf E and its restriction to $\{x\} \times S$.

Now, we will prove the claim in the case $D = \{x\}$ by induction on l_x . We have proven the result for $l_x = 1$. Suppose that it is true for filtrations of length $l_x - 1$. Let $\bar{r}' = \{r_{x,3}, \dots, r_{x,l_x}\}$. Then there exists a closed subscheme

$$\text{FQuot}_{X/S}^{\text{LF}}(F, P, \bar{r}') \subset Q \times_S \text{Grass}(F|_{\{x\} \times S}, r) \times_S \prod_{i=3}^{l_x} \text{Grass}(F|_{\{x\} \times S}, r_{x,i})$$

over Q representing $\mathfrak{F}\text{Quot}_{X/S}^{\text{LF}}(F, P, \bar{r}')$. Let $(E, \{E_{x,1}, E_{x,3}, E_{x,4}, \dots, E_{x,l_x}\})$ be the universal filtered quotient of $FQ' = \text{FQuot}_{X/S}^{\text{LF}}(F, P, \bar{r}')$.

Clearly, parameterizing filtrations $E|_{\{x\} \times T} = E_{x,1} \supseteq \dots \supseteq E_{x,l_x}$ is the same as parameterizing the corresponding subsequent quotients

$$E|_{\{x\} \times T} \twoheadrightarrow E|_{\{x\} \times T} / E|_{x,l_x} \twoheadrightarrow E|_{\{x\} \times T} / E|_{x,l_x-1} \twoheadrightarrow \dots \twoheadrightarrow E|_{\{x\} \times T} / E|_{x,2}$$

Therefore, to give a filtration $E|_{\{x\} \times T} = E_{x,1} \supseteq \dots \supseteq E_{x,l_x}$ is the same as giving a filtration $E|_{\{x\} \times T} = E_{x,1} \supseteq E_{x,3} \supseteq \dots \supseteq E_{x,l_x}$ and a quotient $E|_{\{x\} \times T} / E|_{x,3} = E_{x,1} / E_{x,3} \twoheadrightarrow E|_{\{x\} \times T} / E|_{x,2} = E_{x,1} / E_{x,2}$

Thus, the functor $\mathfrak{F}\text{Quot}_{X/S}^{\text{LF}}(F, P, \bar{r})$ is represented by

$$FQ = FQ' \times_{FQ'} \text{Grass}(E_{x,1} / E_{x,3}, r_{x,2}) = \text{Grass}(E_{x,1} / E_{x,3}, r_{x,2})$$

Let us prove that this product embeds into the desired product of Grassmannians. Let $\pi_{FQ'} : FQ' \rightarrow S$. $E_{x,1} / E_{x,3}$ is a quotient of $\pi_{FQ'}^* F|_{\{x\} \times S}$ over FQ' , so by previous lemma, FQ is a closed subscheme of $\text{Grass}(\pi_{FQ'}^* F|_{\{x\} \times S}, r_{x,2})$ over FQ' . By definition of the Grassmannians functor and the base change formula yields

$$FQ \hookrightarrow \text{Grass}(\pi_{FQ'}^* F|_{\{x\} \times S}, r_{x,2}) \cong FQ' \times_S \text{Grass}(F|_{\{x\} \times S}, r_{x,2})$$

By induction hypothesis, there is a closed embedding over Q

$$FQ' \hookrightarrow Q \times_S \text{Grass}(F|_{\{x\} \times S}, r) \times_S \prod_{i=3}^{l_x} \text{Grass}(F|_{\{x\} \times S}, r_{x,i})$$

so there exists a closed embedding over Q

$$FQ \hookrightarrow Q \times_S \text{Grass}(F|_{\{x\} \times S}, r) \times_S \prod_{i=2}^{l_x} \text{Grass}(F|_{\{x\} \times S}, r_{x,i})$$

Finally, let $D = \{x_1, \dots, x_M\}$ be any finite set of points in C . It is clear that

$$\text{FQuot}_{X/S}^{\text{LF}}(F, P, \{r_{x_j,i}\}) = \text{FQuot}_{X/S}^{\text{LF}}(F, P, \{r_{x_1,i}\}) \times_Q \dots \times_Q \text{FQuot}_{X/S}^{\text{LF}}(F, P, \{r_{x_M,i}\})$$

is a closed subspace of

$$Q \times_S \prod_{x \in D} \text{Grass}(F|_{\{x\} \times S}, r) \times_S \prod_{x \in D} \prod_{i=2}^{l_x} \text{Grass}(F|_{\{x\} \times S}, r_{x,i})$$

over Q which represents the functor $\mathfrak{F}\text{Quot}_{X/S}^{\text{LF}}(F, P, \{r_{x_j,i}\})$. \square

Corollary 3.3.3. *Let F be a coherent sheaf over $X = C \times S$ such that $F|_{\overline{D}}$ is locally free and let $\bar{r} = \{r_{x,i}\}$ for $x \in D$, $1 < i \leq l_x$ be integers. Let $Q \rightarrow \text{Quot}_{X/S}^{\text{LF}}(F, P)$ be any family of isomorphism classes of locally free quotient sheaves of F on X flat over S . Let $\mathfrak{F}\mathfrak{Q}(\bar{r})$ be the functor that associates each S -scheme T the set of isomorphism classes of pairs (E, E_\bullet) consisting on a quotient $\pi^*F \twoheadrightarrow E$ in $Q(T)$ and a filtration by sub-bundles over T*

$$E|_{\{x\} \times T} = E_{x,1} \supsetneq E_{x,2} \supsetneq \dots \supsetneq E_{x,l_x}$$

for each $x \in D$ such that for each $1 < i \leq l_x$, $\text{rk}(E|_{\{x\} \times T}) = r_{x,i}$. Then there is a closed subscheme $FQ(\bar{r})$ of

$$Q \times_S \prod_{x \in D} \text{Grass}(F|_{\{x\} \times S}, r) \times_S \prod_{x \in D} \prod_{i=2}^{l_x} \text{Grass}(F|_{\{x\} \times S}, r_{x,i})$$

over Q representing $\mathfrak{F}\mathfrak{Q}(\bar{r})$.

Proof. The proof is completely analogous to the previous one changing $\text{Quot}_{X/S}^{\text{LF}}(F, P)$ to the given family Q and the universal quotient by its pullback to Q . \square

Grothendieck [Gro61] proved that the quot scheme $\text{Quot}_{X/S}(F, P)$ is a projective scheme over S by constructing an explicit embedding into a certain Grassmannian. More precisely, he stated that there exists an integer M such that for every $m \geq M$ there exists an embedding

$$\psi_m : \text{Quot}_{X/S}(F, P) \hookrightarrow \text{Grass}(H^0(X/S, F(m)), P(m))$$

defined in the following way. By Serre's vanishing theorem, there exists an M such that for every $m \geq M$, $F(m)$ is generated by global sections and $H^0(X/S, F(m))$ is compatible with base change. Grothendieck proved that moreover M can be chosen in a way that for any quotient

$$0 \rightarrow K_G \rightarrow f^*F \rightarrow G \rightarrow 0$$

on $C \times T$, for any T -point of $\text{Quot}_{X/S}(F, P)$ and any $f : T \rightarrow S$, $H^0(C \times T/T, G(m))$ is locally free of rank $P(m)$ and $H^1(C \times T/T, K_G(m)) = 0$. Then, tensoring the previous sequence by $\mathcal{O}_{C \times T}(m)$ and taking the corresponding long exact sequence yields

$$H^0(C \times T/T, f^*F(m)) \rightarrow H^0(C \times T/T, G(m)) \rightarrow H^1(C \times T/T, K_G(m)) \cong 0$$

so we get a T -point of the Grassmannian $\text{Grass}(H^0(X/S, F(m)), P(m))$.

Composing ψ_m with the Plücker embedding

$$\text{Grass}(H^0(X/S, F(m)), P(m)) \hookrightarrow \mathbb{P} \left(\bigwedge^{P(m)} H^0(X/S, F(m)) \right)$$

yields the desired embedding of the quot scheme into a projective bundle over S . We will denote by \mathcal{L}_m the pullback of the corresponding canonical ample line bundle on the Grassmannian by ψ_m .

Theorem 3.3.4. *Let Λ be a sheaf of rings of differential operators on X over S such that $\Lambda|_{\overline{D}}$ is locally free. Let us fix a parabolic type. Let P be a polynomial and let $\{r_{x,i}\}$ be integers for $x \in D$ and $1 < i \leq l_x$. There exists an integer N such that the functor $\mathcal{R}^s : (\text{Sch}_S) \rightarrow (\text{Sets})$ (respectively \mathcal{R}^{ss}) that associates each S -scheme T the set of isomorphism classes of pairs consisting on a (semi-)stable parabolic Λ -module (E, E_\bullet) over $X \times_S T$ with Hilbert polynomial $P_E = P$ such that for each $x \in D$, $\text{rk}(E|_{x \times T}/E_{x,i}) = r_{x,i}$ and an isomorphism*

$$\alpha : \mathcal{O}_T \otimes_{\mathbb{C}} \mathbb{C}^{P(N)} \rightarrow H^0(X \times_S T/T, E(N))$$

is representable by a quasi-projective scheme R^s (R^{ss}) over S .

Proof. Let r be the rank of E . Let N be $|D| + 1$ plus the maximum of the bounds given by Corollary 3.2.6, Corollary 3.2.10 and Lemma 3.2.11. Let Q_5 the subscheme of $\text{Quot}_{X/S}(\Lambda_r \otimes_{\mathcal{O}_X} \mathcal{O}_X(-N) \otimes_{\mathbb{C}} \mathbb{C}^{P(N)}, P)$ described in [Sim94, Theorem 3.8], parameterizing triples (E, φ, α) consisting on a Λ -module E over X with $\varphi : \Lambda \otimes E \rightarrow E$ and an isomorphism $\alpha : \mathcal{O}_S \otimes_{\mathbb{C}} \mathbb{C}^{P(N)} \rightarrow H^0(X/S, E(N))$.

By construction, every sheaf in the family Q_5 is a quotient of $\Lambda_r \otimes \mathcal{O}_X(-N) \otimes_{\mathbb{C}} \mathbb{C}^{P(N)}$. Let Q_5^{LF} be the open subset of triples $(E, \varphi, \alpha) \in Q_5$ such that E is locally free. By the previous corollary, there exists a locally closed subscheme

$$\begin{aligned} FQ_5^{\text{LF}} \hookrightarrow Q_5^{\text{LF}} \times_S \prod_{x \in D} \text{Grass}(\Lambda_r|_{\{x\} \times S} \otimes \mathcal{O}_S(-N) \otimes_{\mathbb{C}} \mathbb{C}^{P(N)}, r) \\ \times_S \prod_{x \in D} \prod_{i=2}^{l_x} \text{Grass}(\Lambda_r|_{\{x\} \times S} \otimes \mathcal{O}_S(-N) \otimes_{\mathbb{C}} \mathbb{C}^{P(N)}, r_{x,i}) \end{aligned}$$

over Q_5^{LF} whose T -points parameterize tuples $(E, E_\bullet, \varphi, \alpha)$ consisting on a rigidified locally free Λ -module (E, φ, α) in $Q_5(T)$ and a filtration by sub-bundles over T

$$E|_{\{x\} \times T} = E_{x,1} \supsetneq E_{x,2} \supsetneq \dots \supsetneq E_{x,l_x}$$

for every $x \in D$ such that for each $1 < i \leq l_x$, $\text{rk}(E|_{\{x\} \times T}/E_{x,i}) = r_{x,i}$.

Let $f : T \rightarrow S$, and let $(E, E_\bullet, \varphi, \alpha)$ be a T -point in FQ_5^{LF} . We say that (E, E_\bullet) satisfies condition R_j (belongs to $R_j(T)$) if for every $x \in X$, $1 < i \leq l_x$ the image of

$$f^*(\Lambda_j) \otimes E_x^i \hookrightarrow f^*(\Lambda_j) \otimes E \twoheadrightarrow E$$

lies in E_x^i . Let $Q_{x,i} = E/E_x^i$. Then the previous condition is equivalent to requiring that the morphism $f^*(\Lambda_j) \otimes E_x^i \rightarrow Q_{x,i}$ given by the composition

$$f^*(\Lambda_j) \otimes E_{x,i} \hookrightarrow f^*(\Lambda_j) \otimes E \twoheadrightarrow E \twoheadrightarrow Q_{x,i}$$

is zero.

Let $(\mathcal{E}, \mathcal{E}_\bullet, \Phi, A)$ be the universal pair for FQ_5^{LF} . For each $x \in D$ and each $i = 1, \dots, l_x$, let \mathcal{E}_x^i be the vector bundle fitting in the short exact sequence

$$0 \longrightarrow \mathcal{E}_x^i \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}|_{\{x\} \times FQ_5^{\text{LF}}} / \mathcal{E}_{x,i} \longrightarrow 0$$

Moreover, take $\mathcal{E}_x^{l_x+1} = \mathcal{E}(-\{x\} \times S)$. Let $Q_{x,i} = \mathcal{E}/\mathcal{E}_x^i$. Let $f : T \rightarrow S$ and let $(E, E_\bullet, \varphi, \alpha)$ be a T -point in FQ_5^{LF} . It is given by the pullback of $(\mathcal{E}, \mathcal{E}_\bullet, \Phi, A)$

by a morphism $e : T \rightarrow FQ_5^{\text{LF}}$. By flatness of $\mathcal{Q}_{x,i}$, $Q_{x,i} = e^* \mathcal{Q}_{x,i}$. Therefore, $(E, E_\bullet, \varphi, \alpha)$ satisfies condition R_j if and only if the pullback by e of the morphisms given by the compositions

$$\pi^*(\Lambda_j) \otimes \mathcal{E}_x^i \hookrightarrow \pi^*(\Lambda_j) \otimes \mathcal{E} \twoheadrightarrow \mathcal{E} \twoheadrightarrow \mathcal{Q}_{x,i}$$

are all zero. By [Yok93, Lemma 4.3], this condition is represented by a closed subscheme R_j of FQ_5^{LF} over S .

Let $R = \bigcap_{j=1}^\infty R_j \subseteq FQ_5^{\text{LF}}$. As Q_5 is Noetherian, Q_5^{LF} is Noetherian. On the other hand, FQ_5^{LF} is quasiprojective over Q_5^{LF} , so it is also Noetherian. Therefore, R is a closed subscheme of FQ_5^{LF} and a point of $FQ_5^{\text{LF}}(T)$ belongs to $R(T)$ if and only if it satisfies the conditions R_j for all $j \geq 1$.

Let R^s (respectively R^{ss}) be the sub-scheme of R parameterizing points of R whose underlying parabolic Λ -module is (semi-)stable. In the next section (Lemma 3.4.7) we will prove that (semi-)stability condition on R is equivalent to GIT-(semi-)stability for a certain group action. Therefore, R^s and R^{ss} are locally closed subschemes of R . Let us prove that R^s and R^{ss} represent the functors \mathcal{R}^s and \mathcal{R}^{ss} respectively.

Let $\Lambda_{R^{ss}}$ be the base change of Λ to R^{ss} via $R^{ss} \rightarrow S$, and let $(\mathcal{E}^{R^{ss}}, \mathcal{E}_\bullet^{R^{ss}}, \varphi^{R^{ss}}, \alpha^{R^{ss}})$ be the universal rigidified parabolic $\Lambda_{R^{ss}}$ -module on R^{ss} . As $(\mathcal{E}^{R^{ss}}, \varphi^{R^{ss}}, \alpha^{R^{ss}})$ is a R^{ss} -point of Q_5^{LF} , we have a natural morphism

$$\alpha^{R^{ss}} : \mathcal{O}_{R^{ss}} \otimes_{\mathbb{C}} \mathbb{C}^{P(N)} \rightarrow H^0(C \times R^{ss}/R^{ss}, \mathcal{E}^{R^{ss}}(N))$$

As N was chosen so that the conclusion of Corollary 3.2.6 holds and the restriction of $(\mathcal{E}^{R^{ss}}, \mathcal{E}_\bullet^{R^{ss}}, \varphi^{R^{ss}})$ to any closed point is semistable, $H^0(C \times R^{ss}/R^{ss}, \mathcal{E}^{R^{ss}}(N))$ is locally free of rank $P(N)$ and compatible with base change. On the other hand, condition Q_2 in Simpson's construction of scheme Q_5 [Sim94, Theorem 3.8] imply that $\alpha^{R^{ss}}$ is injective on the fibers over closed points, so it is an isomorphism. Therefore, we obtain a universal object $(\mathcal{E}^{R^{ss}}, \mathcal{E}_\bullet^{R^{ss}}, \varphi^{R^{ss}}, \alpha^{R^{ss}})$ over R^{ss} , inducing the quadruples described by the functor \mathcal{R}^{ss} for every base change $e : T \rightarrow R^{ss}$. Let us verify that R^{ss} represents the functor \mathcal{R}^{ss} .

Let $f : T \rightarrow S$ be an S -scheme of finite type, and let $(E, E_\bullet, \varphi, \alpha)$ be a pair of a semistable parabolic Λ -module (E, E_\bullet, φ) over $C \times T$ with Hilbert polynomial P and the given fixed parabolic structure and an isomorphism

$$\alpha : \mathcal{O}_T \otimes_{\mathbb{C}} \mathbb{C}^{P(N)} \rightarrow H^0(C \times T/T, E(N))$$

By Corollary 3.2.6, $E(N)$ is generated by global sections and we have a surjection

$$\mathcal{O}_{C \times T}(-N) \otimes_{\mathcal{O}_T} H^0(C \times T/T, E(N)) \rightarrow E \rightarrow 0$$

Therefore, α induces a morphism

$$f^*(\Lambda_r) \otimes_{\mathcal{O}_{C \times T}} (-N) \otimes_{\mathbb{C}} \mathbb{C}^{P(N)} \cong f^*(\Lambda_r) \otimes_{\mathcal{O}_{C \times T}} (-N) \otimes H^0(C \times T/T, E(N)) \rightarrow E \rightarrow 0$$

which defines a point on Q_5^{LF} clearly. Restricting the previous morphism to $\{x\} \times T$, for $x \in D$, we get quotients

$$f^*(\Lambda_r)|_{\{x\} \times T} \otimes_{\mathcal{O}_T} (-N) \otimes_{\mathbb{C}} \mathbb{C}^{P(N)} \rightarrow E|_{\{x\} \times T} = E_{x,1} \rightarrow 0$$

The rest of the parabolic structure E_\bullet induces a set of quotients

$$f^*(\Lambda_r)|_{\{x\} \times T} \otimes \mathcal{O}_T(-N) \otimes_{\mathbb{C}} \mathbb{C}^{P(N)} \rightarrow E|_{\{x\} \times T} = E_{x,1} \rightarrow E_{x,1}/E_{x,i} \rightarrow 0$$

for every $x \in D$ and $1 < i \leq l_x$, so $(E, E_\bullet, \varphi, \alpha)$ defines clearly a point in $FQ_5(T)$. As the filtration is preserved by $f^*\Lambda$, it lies in $R(T)$. The parabolic Λ -module is semistable, so it represents a point $e \in R^{ss}(T)$. By universality of $(\mathcal{E}^{R^{ss}}, \mathcal{E}_\bullet^{R^{ss}}, \varphi^{R^{ss}}, \alpha^{R^{ss}})$ yields $e^*(\mathcal{E}^{R^{ss}}, \mathcal{E}_\bullet^{R^{ss}}, \varphi^{R^{ss}}, \alpha^{R^{ss}}) \cong (E, E_\bullet, \varphi, \alpha)$.

The previous construction is clearly compatible with base change, so it defines a natural transformation $\mathcal{R}^{ss} \rightarrow \text{Hom}(\cdot, R^{ss})$. Taking the pullback of the universal object defines an inverse natural transformation $\text{Hom}(\cdot, R^{ss}) \rightarrow \mathcal{R}^{ss}$, so R^{ss} represents \mathcal{R}^{ss} .

By definition, the points in R^s represent points $(E, E_\bullet, \varphi, \alpha)$ in R^{ss} such that (E, E_\bullet, φ) is stable. Then, the restriction of the natural transformation $\text{Hom}(\cdot, R^{ss}) \rightarrow \mathcal{R}^{ss}$ to $\text{Hom}(\cdot, R^s) \subseteq \text{Hom}(\cdot, R^{ss})$ lies in the subfunctor $\mathcal{R}^s \subseteq \mathcal{R}^{ss}$. As the natural transformation is an isomorphism, its restriction to $\text{Hom}(\cdot, R^s)$ is an isomorphism onto its image, so we get an isomorphism of functors $\text{Hom}(\cdot, R^s) \rightarrow \mathcal{R}^s$. Then R^s represents \mathcal{R}^s and by [Sim94, Lemma 1.11], R^s is an open subscheme of R^{ss} . \square

3.4 Geometric invariant theory

Two different T -points of the previous scheme R^s (R^{ss}) with the same underlying parabolic Λ -module (E, E_\bullet, φ) differ only in the choice of the isomorphism

$$\alpha : \mathcal{O}_T \otimes_{\mathbb{C}} \mathbb{C}^{P(N)} \rightarrow H^0(X \times_S T/T, E(N))$$

Therefore, they are related by an automorphism of $\mathcal{O}_T \otimes_{\mathbb{C}} \mathbb{C}^{P(N)}$, which is equivalent to a morphism $T \rightarrow \text{GL}_{P(N)}(\mathbb{C})$. As dilations $T \rightarrow \mathbb{C}^*$ preserve the isomorphism class of (E, E_\bullet, φ) up to tensoring by a line bundle over the parameter space $L \rightarrow T$, two isomorphism classes of T -families of parabolic Λ -modules differ effectively by a morphism $T \rightarrow \text{SL}_{P(N)}(\mathbb{C})$.

Therefore, the moduli functor of (semi-)stable parabolic Λ -modules is clearly a categorical quotient of the functor described in the previous theorem by the action of $\text{SL}_{P(N)}(\mathbb{C})$ on $\mathbb{C}^{P(N)}$. Subsequently, we can obtain a coarse moduli space for the desired moduli functor by finding a good categorical quotient of the scheme R^s (R^{ss}) described in the previous theorem by the action of $\text{SL}_{P(N)}(\mathbb{C})$. We will use Geometric Invariant Theory to describe this quotient.

First of all, we will briefly review Mumford's notation on GIT quotients and the main GIT-stability theorem. Let X be a proper complex algebraic scheme and let G be an algebraic group acting on X . Let λ be a one parameter subgroup (1-PS) of G . For every closed point $x \in X$, composing with the action of G on X yields a morphism of \mathbb{G}_m to X . By properness of X , it extends to a morphism $f_{x,\lambda} : \mathbb{A}^1 \rightarrow X$ such that $f_{x,\lambda}(0)$ is a fixed point for the \mathbb{G}_m action. Let L be a G -linearized line bundle over X . By [Mum82, §1.3], the induced \mathbb{G}_m linearization of L restricted to $f_{x,\lambda}(0)$ is given by a character of \mathbb{G}_m , $\xi(\alpha) = \alpha^r$ for $\alpha \in \mathbb{G}_m$. We define

$$\mu_G^L(x, \lambda) = -r$$

We will be interested in the following two functorial properties of μ

1. For fixed x and λ , $\mu_G^{(\bullet)}(x, \lambda)$ defines a homomorphism from the group of G -linearized line bundles on X to \mathbb{Z} .
2. If $X \rightarrow Y$ is a G -linear morphism of schemes on which G acts and L is a G -linearized line bundle over Y then f^*L is a G -linearized line bundle over X and

$$\mu_G^{f^*L}(x, \lambda) = \mu_G^L(f(x), \lambda)$$

Lemma 3.4.1. *Let $\varepsilon_1, \dots, \varepsilon_M$ be positive rational numbers. Let G be an algebraic group and let X_1, \dots, X_M be complex projective schemes such that for each $i = 1, \dots, M$, G acts on X_i . For each i let $\mathcal{O}_{X_i}(1)$ be an ample G -linearized line bundle over X_i . Then for each closed point $x = (x_1, \dots, x_M)$ in $X_1 \times \dots \times X_M$ and each 1-PS λ of G*

$$\mu_G^{\bigotimes_{i=1}^M \mathcal{O}_{X_i}(\varepsilon_i)}(x, \lambda) = \sum_{i=1}^M \varepsilon_i \mu_G^{\mathcal{O}_{X_i}(1)}(x_i, \lambda)$$

Proof. See [Mum82, Chapter 3]. □

Lemma 3.4.2. *Let G be an algebraic group acting on a complex proper scheme X , and let H be a subgroup of G . Let L be any G -linearized line bundle over X . Then H acts on X , the G -linearization of L induces an H -linearization of L . Let λ be a 1-PS of H , and let $\bar{\lambda}$ be the 1-PS of G obtained composing λ with the inclusion of H in G . Then for every geometric point x*

$$\mu_H^L(x, \lambda) = \mu_G^L(x, \bar{\lambda})$$

Proof. For every closed point $x \in X$, the composition of the action of H on X with λ coincides with the composition of the action of G with $\bar{\lambda}$. Therefore, both actions induce the same morphism $f_{x, \lambda} : \mathbb{A}^1 \rightarrow X$ and the same \mathbb{G}_m -linearization of L . Thus, the linearization of L is given by the same character and the equality holds by definition of μ . □

Theorem 3.4.3 ([Mum82, Theorem 2.1]). *Let G be a reductive group acting on a proper complex scheme X . Let L be a G -invariant ample line bundle over X . Then for every geometric point of X*

1. x is GIT-semistable if and only if $\mu^L(x, \lambda) \geq 0$ for all 1-PS λ .
2. x is GIT-stable if and only if $\mu^L(x, \lambda) > 0$ for all 1-PS λ .

Now, we will apply the previous Theorem to compute the GIT-stability condition for the linear action on a product of Grassmannians.

Let V and W_i be complex vector spaces for $i = 1, \dots, M$. Let p_i be an integer $0 \leq p_i \leq \dim W_i$ for $i = 1, \dots, M$. For every i , let $\text{Grass}(p_i, W_i \otimes V)$ and $\text{Grass}(W_i \otimes V, p_i)$ denote the Grassmannians of subspaces and quotients respectively. There is a canonical isomorphism

$$\text{Grass}(W_i \otimes V, p_i) \cong \text{Grass}(\dim(W_i) \dim(V) - p_i, W_i \otimes V)$$

Let us consider the canonical action of $\text{SL}(V)$ on $\text{Grass}(W_i \otimes V, p_i)$ extended from the action on V . For each i , $\text{Grass}(W_i \otimes V, p_i)$ gets embedded into $\mathbb{P}(\bigwedge^{p_i}(W_i \otimes V))$ by Plücker embedding. For each i , let $\mathcal{O}_i(1)$ denote the pullback of $\mathcal{O}_{\mathbb{P}(\bigwedge^{p_i}(W_i \otimes V))}(1)$.

Lemma 3.4.4. *Let $\varepsilon_1, \dots, \varepsilon_M$ be positive rational numbers. Let $x = (L_1, \dots, L_M)$ be a geometric point of $\prod_{i=1}^M \text{Grass}(p_i, W_i \otimes V)$, i.e., let L_i be a subspace of $W_i \otimes V$ of dimension p_i for each $i = 1, \dots, M$. Then x is GIT-(semi-)stable with respect to the action of $\text{SL}(V)$, linearized by $\Theta = \bigotimes_{i=1}^M \mathcal{O}_i(\varepsilon_i)$, if and only if for all linear subspaces $L \subseteq V$*

$$\frac{\sum_{i=1}^M \varepsilon_i \dim(L_i \cap (W_i \otimes L))}{\dim L} (\leq) < \frac{\sum_{i=1}^M \varepsilon_i p_i}{\dim V} \quad (3.4.1)$$

Proof. Let $n = \dim(V)$ and $m_i = \dim(W_i)$. Let λ be any 1-PS of $\text{SL}(V)$. Let $\{e_1, \dots, e_n\}$ be a basis of V such that for every $t \in \mathbb{C}$, the matrix of $\lambda(t)$ in the basis $\{e_1, \dots, e_n\}$ is given by

$$\lambda(t) = (t^{r_i} \delta_{ij})_{i,j=1}^n$$

Where $r_1 \geq r_2 \geq \dots \geq r_n$ and $\sum_{i=1}^n r_i = 0$. Then, fixed any basis $\{w_{i,1}, \dots, w_{i,m_i}\}$ of W_i , $\{w_{i,k} \otimes e_j\}$ is a basis of $W_i \otimes V$. Let $\bar{\lambda}$ be the composition of λ with the canonical inclusion of $\text{SL}(V)$ in $\text{SL}(W_i \otimes V)$. Under, the given adapted basis, $\bar{\lambda}(t)$ has the form

$$\bar{\lambda}(t) = \begin{pmatrix} \overbrace{\begin{matrix} t^{r_1} & & \\ & \ddots & \\ & & t^{r_1} \end{matrix}}^{m_i} & & \\ & \ddots & \\ & & \underbrace{\begin{matrix} t^{r_n} & & \\ & \ddots & \\ & & t^{r_n} \end{matrix}}^{m_i} \end{pmatrix}$$

Let us call \bar{r}_l to the exponent of t for the l -th entry on the diagonal of $\bar{\lambda}(t)$, i.e., for $1 \leq j \leq n$ and $1 \leq k \leq m_i$

$$\bar{r}_{(j-1)m_i+k} = r_j = r_{\lfloor l/m_i \rfloor}$$

On the other hand, for each subset $I \subseteq \{1, \dots, n\} \times \{1, \dots, m_i\}$, let \bar{L}_I denote the linear subspace of $V \times W_i$ defined as

$$\bar{L}_I = \left\{ \sum_{j=1}^n \sum_{k=1}^{\dim(W_i)} x_{j,k} w_{i,k} \otimes e_j \mid x_{j,k} = 0 \forall (j,k) \in I \right\}$$

Similarly, for each subset $I \subseteq \{1, \dots, n\}$, let L_I be the subspace of V defined as

$$L_I = \left\{ \sum_{j=1}^n x_j e_j \mid x_j = 0 \forall j \in I \right\}$$

Finally, if we denote $I_l = \{(j,k) \in \{1, \dots, n\} \times \{1, \dots, m_i\} \mid (j-1)m_i + k \geq l\}$, for $1 \leq l \leq nm_i$, combining Lemma 3.4.2 and [Mum82, Proposition 4.3] yields

$$\mu_{\text{SL}(V)}^{\mathcal{O}_i(1)}(L_i, \lambda) = \mu_{\text{SL}(W_i \otimes V)}^{\mathcal{O}_i(1)}(L_i, \bar{\lambda}) = -p_i \bar{r}_{nm_i} + \sum_{l=1}^{nm_i-1} \dim(L_i \cap \bar{L}_{I_{l+1}}) (\bar{r}_{l+1} - \bar{r}_l)$$

On the other hand, for each l , $\bar{r}_{l+1} - \bar{r}_l = r_{\lceil(l+1)/m_i\rceil} - r_{\lceil l/m_i\rceil}$ is only nonzero if l is a multiple of m_i . Moreover, for each $1 \leq j < n$,

$$L_{I_{jm_i+1}} = \mathcal{L}(\{e_{j'} \otimes w_{i,k} | 1 \leq j' \leq j, 1 \leq k \leq m_i\}) = \\ W_i \otimes \mathcal{L}(\{e_{j'} | 1 \leq j' \leq j\}) = W_i \otimes L_{\{j+1, \dots, n\}}$$

Therefore

$$\mu_{\text{SL}(V)}^{\mathcal{O}_i(1)}(L_i, \lambda) = -p_i r_n + \sum_{j=1}^{n-1} \dim(L_i \cap \bar{L}_{I_{jm_i+1}}) (r_{j+1} - r_j) = \\ -p_i r_n + \sum_{j=1}^{n-1} \dim(L_i \cap (W_i \otimes L_{\{j+1, \dots, n\}})) (r_{j+1} - r_j)$$

Now, by Lemma 3.4.1,

$$\mu_{\text{SL}(V)}^{\Theta}(x, \lambda) = -\sum_{i=1}^M \varepsilon_i p_i r_n + \sum_{i=1}^M \varepsilon_i \sum_{j=1}^{n-1} \dim(L_i \cap (W_i \otimes L_{\{j+1, \dots, n\}})) (r_{j+1} - r_j)$$

By Theorem 3.4.3, x is GIT-(semi)-stable if and only if $\mu_{\text{SL}(V)}^{\Theta}(x, \lambda)$ is positive (non-negative) for all 1-PS λ . $\mu_{\text{SL}(V)}^{\Theta}(x, \lambda)$ is a linear function of the r_j . Thus, its value is positive (non-negative) for all r_j such that $r_1 \geq \dots \geq r_n$ and $\sum_{j=1}^n r_j = 0$ if and only if it is positive (non-negative) for the extreme sets of r_j

$$\begin{cases} r_1 = r_2 = \dots = r_l = n - l \\ r_{l+1} = \dots = r_n = -l \end{cases}$$

for every $1 \leq l < n$. For such $\{r_j\}$, $r_{j+1} - r_j$ is nonzero just for $j = l$, so $\mu_{\text{SL}(V)}^{\Theta}(x, \lambda)(\geq) > 0$ for such r_j if and only if

$$l \sum_{i=1}^M \varepsilon_i p_i - \sum_{i=1}^M \varepsilon_i \dim(L_i \cap (W_i \otimes L_{\{l+1, \dots, n\}})) n(\geq) > 0 \quad (3.4.2)$$

Every 1-PS of $\text{SL}(V)$ is conjugate to a 1-PS which is diagonalized in the basis $\{e_1, \dots, e_n\}$ and for which $r_1 \geq r_2 \geq \dots \geq r_n$. Therefore, x is GIT-semistable if and only if condition (3.4.2) holds for every basis choice $\{e_1, \dots, e_n\}$. Under base change, $L_{\{l+1, \dots, n\}}$ ranges over the set of linear subspaces of V of dimension l , so x is GIT-(semi)-stable if and only if for every subspace $L \subsetneq V$

$$\dim(L) \sum_{i=1}^M \varepsilon_i p_i - n \sum_{i=1}^M \varepsilon_i \dim(L_i \cap (W_i \otimes L)) (\geq) > 0$$

□

Corollary 3.4.5. *Let $\varepsilon_1, \dots, \varepsilon_M$ be rational numbers. Let $x = (\varphi_1, \dots, \varphi_M)$ be a geometric point of $\prod_{i=1}^M \text{Grass}(W_i \otimes V, p_i)$, i.e., let $\varphi_i : W_i \otimes V \twoheadrightarrow L_i$ be a quotient of dimension p_i for each $i = 1, \dots, M$. Then x is GIT-(semi)-stable with respect to $\bigotimes_{i=1}^m \mathcal{O}_i(\varepsilon_i)$ if and only if for all linear subspaces $L \subseteq V$*

$$\frac{\sum_{i=1}^M \varepsilon_i \dim(\varphi_i(W_i \otimes L))}{\dim L} (\geq) > \frac{\sum_{i=1}^M \varepsilon_i p_i}{\dim V}$$

Proof. Let $p'_i = \dim(V) \dim(W_i) - p_i$. For each $\varphi_i : W_i \otimes V \rightarrow L_i$, let $L'_i = \text{Ker}(\varphi_i)$ be the corresponding image of φ_i in $\text{Grass}(p'_i, W_i \otimes V)$, and let $x' = (L'_1, \dots, L'_M)$. Clearly, x is GIT-semistable if and only if x' is GIT-semistable, i.e., if and only if inequality (3.4.1) holds for every linear subspace $L \subseteq V$. Let $L \subseteq V$ be any linear subspace. Then, for each i

$$\begin{aligned} \dim(L'_i \cap (W_i \otimes L)) &= \dim(\text{Ker}(\varphi_i) \cap (W_i \otimes L)) = \dim(\text{Ker}(\varphi_i|_{W_i \otimes L})) = \\ &= \dim(W_i \otimes L) - \dim(\text{Im}(\varphi_i|_{W_i \otimes L})) = \dim(W_i) \dim(L) - \dim(\varphi_i(W_i \otimes L)) \end{aligned}$$

Therefore, by the previous Lemma, x' is (semi-)stable if and only if

$$\begin{aligned} \frac{\sum_{i=1}^M \varepsilon_i \dim(\varphi_i(W_i \otimes L))}{\dim L} &= \frac{\sum_{i=1}^M \varepsilon_i (\dim(W_i) \dim(L) - \dim(L_i \cap (W_i \otimes L)))}{\dim L} = \\ \sum_{i=1}^M \varepsilon_i \dim(W_i) - \frac{\sum_{i=1}^M \varepsilon_i \dim(L_i \cap (W_i \otimes L))}{\dim L} &(\geq) > \sum_{i=1}^M \varepsilon_i \dim(W_i) - \frac{\sum_{i=1}^M \varepsilon_i p'_i}{\dim(V)} = \\ &= \frac{\sum_{i=1}^M \varepsilon_i (\dim(W_i) \dim(V) - p'_i)}{\dim(V)} = \frac{\sum_{i=1}^M \varepsilon_i p_i}{\dim(V)} \end{aligned}$$

□

We can extend the previous GIT-stability conditions for Grassmannians of locally free vector bundles by means of the following Lemma due to Simpson

Lemma 3.4.6 ([Sim94, Lemma 1.13]). *Suppose that $Z \rightarrow S$ is a projective scheme. Let G be a reductive algebraic group acting on Z , such that the action is trivial on S and preserves the morphism $Z \rightarrow S$. Let \mathcal{L} be a relatively very ample linearizable invertible sheaf for the action of G . If $t \rightarrow S$ is a geometric point then the (semi-)stable points of the fiber Z_s are those which are (semi-)stable in the total space, i.e. $Z_t^s = (Z^s)_t$ ($Z_t^{ss} = (Z^{ss})_t$ respectively).*

Let \mathcal{W}_i be a locally free vector bundle on S for $i = 1, \dots, l$. Let V be a finite dimensional vector space. Let r_i be integers and let ε_i be rational numbers for $i = 1, \dots, l$. Let $G_i = \text{Grass}(\mathcal{W}_i \otimes_{\mathbb{C}} V, r_i) \rightarrow S$ and let us consider the canonical action of $\text{SL}(V)$ on G_i for each i . Let $\mathcal{O}_i(1)$ denote the canonical G -linearizable line bundle on S corresponding to the Plücker embedding

$$\text{Grass}(\mathcal{W}_i \otimes_{\mathbb{C}} V, r_i) \rightarrow \mathbb{P} \left(\bigwedge^{r_i} \mathcal{W}_i \otimes_{\mathbb{C}} V \right)$$

Then $\text{SL}(V)$ acts on $G = \prod_{i=1}^l G_i$ and $\Theta = \bigotimes_{i=1}^l \mathcal{O}_i(\varepsilon_i)$ is a relatively very ample G -linearizable invertible sheaf, flat over S . By the previous Lemma, a geometric point in G , standing over a fiber $t \rightarrow S$ is (semi-)stable if and only if the corresponding point in $G_s = \prod_{i=1}^l \text{Grass}(\mathcal{W}_i|_s \otimes V, r_i)$ satisfies the condition of Corollary 3.4.5.

Lemma 3.4.7. *Let R be the quasi-projective scheme described by Theorem 3.3.4. Let $\mathcal{O}_{x,i}(1)$ denote the canonical ample line bundle on $\text{Grass}(\Lambda_r|_{\{x\} \times S} \otimes \mathcal{O}_S(-N) \otimes_{\mathbb{C}} \mathbb{C}^{P(N)}, r_{x,i})$ under the Plücker embedding for $x \in D$ and $1 \leq i \leq l_x$, where we are setting $r_{x,1} = r$ for each $x \in D$. There exists an integer \bar{m} such that for every*

$m \geq \bar{m}$ there exist positive rational numbers $\varepsilon_0, \varepsilon_{x,i}$, for $x \in D, i = 1, \dots, l_x$ such that a geometric point of R is GIT-(semi-)stable for the linearization induced by the restriction of

$$\Theta = \mathcal{L}_m(\varepsilon_0) \otimes \bigotimes_{x \in D} \bigotimes_{i=1}^{l_x} \mathcal{O}_{x,i}(\varepsilon_i)$$

to R if and only if it corresponds to a (semi-)stable parabolic Λ -module.

Proof. Let $M = |D|$. For each $x \in D$, let $\varepsilon_{x,i} = \alpha_{x,i} - \alpha_{x,i-1}$ for $1 < i \leq l_x$ and $\varepsilon_{x,1} = 1 - \alpha_{x,l_x}$. As $\alpha_{x,i}$ are strictly crescent and less than 1, it is clear that $\varepsilon_{x,i} > 0$ for all $x \in D$ and $i = 1, \dots, l_x$. For any $m > N$ let

$$\varepsilon_0 = \frac{\text{par-}\mu(E) + N - M + 1 - g}{m - N}$$

The choice of N in Theorem 3.3.4 ensures that $\varepsilon_0 > 0$. By definition a Λ -module over $C \times S$ is (semi-)stable if and only if its restriction to X_s for every $s \in S$ is (semi-)stable. On the other hand, by the previous lemma an S -point in R is (semi-)stable, if and only if its specification to every $s \in S$ is (semi-)stable. Therefore, we can restrict ourselves to closed points of R , i.e., Λ -modules over a certain geometric fiber X_s of $C \times S$. First, let us prove that all GIT-semistable points of R are semistable. Let (E, E_\bullet, φ) be a parabolic Λ -module over X_s , for some $s \in S$ underlying a GIT-semistable point of R . Suppose that (E, E_\bullet, φ) is unstable. Let (F, F_\bullet, φ) be the maximum destabilizing sub- Λ -module. By maximality of the parabolic slope, it is a semistable parabolic Λ -module and we can assume without loss of generality that F has the induced parabolic structure from (E, E_\bullet) .

By Corollary 3.2.10, $F(N)$ is generated by global sections. Let $L = H^0(X_s, F(N))$. By Corollary 3.4.5, GIT-semi-stability of $(E, E_\bullet, \varphi, \alpha)$ for the linearization Θ implies that

$$\begin{aligned} & \frac{\varepsilon_0 \dim \text{Im}(H^0(X_s, \Lambda_{X_s,r} \otimes \mathcal{O}_{X_s}(m - N)) \otimes L \rightarrow H^0(X_s, E(m)))}{\dim L} \\ & + \frac{\sum_{x \in X} \varepsilon_{x,1} \dim \text{Im}(\Lambda_r|_{\{(x,s)\}} \otimes L \rightarrow E_{x,1})}{\dim L} \\ & + \frac{\sum_{x \in D} \sum_{i=2}^{l_x} \varepsilon_{x,i} \dim \text{Im}(\Lambda_r|_{\{(x,s)\}} \otimes \mathcal{O}_S(-N)|_s \otimes L \rightarrow E_{x,1}/E_{x,i})}{\dim L} \\ & \geq \frac{\varepsilon_0 P_E(m) + \sum_{x \in D} (\varepsilon_{x,1} r + \sum_{i=2}^{l_x} \varepsilon_{x,i} r_{x,i})}{P_E(N)} \quad (3.4.3) \end{aligned}$$

On the other hand, by N -regularity of F , yields

$$\begin{array}{ccc} H^0(X_s, \Lambda_{X_s,r} \otimes \mathcal{O}_{X_s}(m - N)) \otimes H^0(X_s, E(N)) & \longrightarrow & H^0(X_s, E(m)) \\ \uparrow & & \uparrow \\ H^0(X_s, \Lambda_{X_s,r} \otimes \mathcal{O}_{X_s}(m - N)) \otimes H^0(X_s, F(N)) & \longrightarrow & H^0(X_s, F(m)) \end{array}$$

Thus

$$\dim \text{Im}(H^0(X_s, \Lambda_{X_s,r} \otimes \mathcal{O}_{X_s}(m - N)) \otimes L \rightarrow H^0(X_s, E(m))) = P_F(m)$$

As $\Lambda_{X_s, r} \otimes \mathcal{O}_{X_s}(-N) \otimes L$ generates F , $\Lambda_r|_{\{(x, s)\}} \otimes \mathcal{O}_S(-N)|_s \otimes L$ generates $F|_{\{(x, s)\}} = F_{x,1}$ in $E_{x,1}$ for each $x \in D$. Therefore, for each $x \in D$ and each $1 < i \leq l_x$

$$\begin{aligned} \dim \operatorname{Im} (\Lambda_r|_{\{(x, s)\}} \otimes \mathcal{O}_S(-N)|_s \otimes L \rightarrow (E_{x,1}/E_{x,i})) &= \dim \left(\frac{F|_{\{(x, s)\}}}{E_{x,i}} \right) \\ &= \operatorname{rk} \left(\frac{F_{x,1}}{E_{x,i} \cap F_{x,1}} \right) = \operatorname{rk} \left(\frac{F_{x,1}}{F_{x,i}} \right) = \operatorname{rk}(F_{x,1}) - \operatorname{rk}(F_{x,i}) = \operatorname{rk}(F) - \operatorname{rk}(F_{x,i}) \end{aligned}$$

Therefore, substituting the previous computations in equation (3.4.3) and taking into account that by Corollary 3.2.10, $\dim L = h^0(X_s, F(N)) = P_F(N)$, yields

$$\begin{aligned} \frac{\varepsilon_0 P_F(m) + \sum_{x \in D} \left(\varepsilon_{x,1} \operatorname{rk}(F) + \sum_{i=2}^{l_x} \varepsilon_{x,i} (\operatorname{rk}(F) - \operatorname{rk}(F_{x,i})) \right)}{P_F(N)} \\ \geq \frac{\varepsilon_0 P_E(m) + \sum_{x \in D} \left(\varepsilon_{x,1} r + \sum_{i=2}^{l_x} \varepsilon_{x,i} r_{x,i} \right)}{P_E(N)} \quad (3.4.4) \end{aligned}$$

For each $x \in D$ we have

$$\begin{aligned} \varepsilon_{x,1} \operatorname{rk}(F) + \sum_{i=2}^{l_x} \varepsilon_{x,i} (\operatorname{rk}(F) - \operatorname{rk}(F_{x,i})) &= \sum_{i=1}^{l_x} \varepsilon_{x,i} \operatorname{rk}(F) - \sum_{i=2}^{l_x} (\alpha_{x,i} - \alpha_{x,i-1}) \operatorname{rk}(F_{x,i}) \\ &= (1 - \alpha_1) \operatorname{rk}(F) + \alpha_1 \operatorname{rk}(F_{x,2}) - \sum_{i=2}^{l_x-1} \alpha_{x,i} (\operatorname{rk}(F_{x,i}) - \operatorname{rk}(F_{x,i+1})) - \alpha_{x,l_x} \operatorname{rk}(F_{x,l_x}) \\ &= \operatorname{rk}(F) - \sum_{i=1}^{l_x-1} \alpha_{x,i} (\operatorname{rk}(F_{x,i}) - \operatorname{rk}(F_{x,i+1})) - \alpha_{x,l_x} \operatorname{rk}(F_{x,l_x}) = \operatorname{rk}(F) - \operatorname{wt}_x(F) \end{aligned} \quad (3.4.5)$$

Adding up over x and substituting in both sides of equation (3.4.4) yields

$$\frac{\varepsilon_0 P_F(m) + M \operatorname{rk}(F) - \operatorname{wt}(F)}{P_F(N)} \geq \frac{\varepsilon_0 P_E(m) + M \operatorname{rk}(E) - \operatorname{wt}(E)}{P_E(N)}$$

By Riemann-Roch formula, $P_F(k) = \operatorname{rk}(F)(k+1-g) + \deg(F)$. Substituting in the previous equation and dividing both numerators and denominators by the corresponding ranks yields

$$\frac{\varepsilon_0(m+1-g) + M + \varepsilon_0 \mu(F) - \eta(F)}{N+1-g+\mu(F)} \geq \frac{\varepsilon_0(m+1-g) + M + \varepsilon_0 \mu(E) - \eta(E)}{N+1-g+\mu(E)}$$

Subtracting ε_0 to both sides of the inequality gives

$$\frac{\varepsilon_0(m-N) + M - \eta(F)}{N+1-g+\mu(F)} \geq \frac{\varepsilon_0(m-N) + M - \eta(E)}{N+1-g+\mu(E)}$$

By the choice of N , both denominators are positive, so by multiplying and grouping one obtains

$$\begin{aligned} \mu(F) (\varepsilon_0(m-N) + M - \eta(E)) + \eta(F) (N+1-g+\eta(E)) \\ \leq \mu(E) (\varepsilon_0(m-N) + M) + \eta(E) (N+1-g) \end{aligned}$$

Adding and subtracting $\mu(E)\eta(E)$ to the right hand side of the inequality yields

$$\begin{aligned} \mu(F)(\varepsilon_0(m-N) + M - \eta(E)) + \eta(F)(N+1-g+\mu(E)) \\ \leq \mu(E)(\varepsilon_0(m-N) + M - \eta(E)) + \eta(E)(N+1-g+\mu(E)) \end{aligned}$$

By the choice of ε_0 , $\varepsilon_0(m-N) + M - \eta(E) = N+1-g+\mu(E) > 0$. Thus

$$\mu(F) + \eta(F) \leq \mu(E) + \eta(E)$$

contradicting that F destabilizes E .

Moreover $\text{par-}\mu(F) = \text{par-}\mu(E)$ if and only if (3.4.4) is an equality. Therefore, if (E, E_\bullet, φ) is a strictly semistable parabolic Λ -module and (F, F_\bullet, φ) is a parabolic sub- Λ -module such that $\text{par-}\mu(F) = \text{par-}\mu(E)$, then taking $L = H^0(X_s, F(N))$ we obtain equality in (3.4.4) and, therefore, $(E, E_\bullet, \varphi, \alpha)$ is strictly GIT semistable.

Now, we will prove that semi-stability implies GIT-stability for big enough m . Let (E, E_\bullet) be a semistable parabolic Λ_s -module over X_s . By Corollary 3.2.10, the image of (E, E_\bullet) under the embedding ψ_m is GIT-semistable if and only if condition (3.4.3) holds for every $L \subseteq \mathbb{C}^{P(N)}$.

Let $L \subseteq \mathbb{C}^{P(N)}$ be any vector subspace. Let F be the subsheaf of E obtained as the image of L under the quotient $\Lambda_{X_s, r} \otimes \mathcal{O}_{X_s}(-N) \otimes_{\mathbb{C}} \mathbb{C}^{P(N)} \rightarrow E$ and let $K_{L, E}$ be the kernel of the quotient

$$0 \rightarrow K_{L, E} \rightarrow \Lambda_{X_s, r} \otimes \mathcal{O}_{X_s}(-N) \otimes_{\mathbb{C}} L \rightarrow F \rightarrow 0$$

Tensoring the previous short exact sequence by $\mathcal{O}_X(m)$ yields

$$0 \rightarrow K_{L, E}(m) \rightarrow \Lambda_{X_s, r} \otimes \mathcal{O}_{X_s}(m-N) \otimes_{\mathbb{C}} L \rightarrow F(m) \rightarrow 0$$

The set of possible subsheaves F generated this way and the set of possible kernels $K_{L, E}$ both form bounded families of sheaves on X flat over S , so by [Sim94, Lemma 1.9], there exists an m_0 such that for every $m \geq m_0$, $H^1(X_s, K_{L, E}(m)) = 0$ and $H^1(X_s, F(m)) = 0$. Therefore, the corresponding long exact sequence in cohomology reduces to

$$\begin{aligned} 0 \rightarrow H^0(X_s, K_{L, E}(m)) \rightarrow H^0(X_s, \Lambda_{X_s, r} \otimes \mathcal{O}_{X_s}(m-N) \otimes_{\mathbb{C}} L \\ \rightarrow H^0(X_s, F(m)) \rightarrow H^1(X_s, K_{L, E}(m)) = 0 \end{aligned}$$

Thus,

$$\dim \text{Im}(H^0(X_s, \Lambda_{X_s, r} \otimes \mathcal{O}_{X_s}(m-N) \otimes L \rightarrow H^0(X_s, E(m))) = h^0(X_s, F(m)) = P_F(m)$$

On the other hand, similarly to the first part of the proof, $L \otimes \Lambda_r|_{\{(x, s)\}} \otimes \mathcal{O}_S(-N)|_s$ generates $F_{x, 1}$ in $E_{x, 1}$ for each $x \in D$. Therefore, for each $x \in D$ and each $1 < i \leq l_x$

$$\begin{aligned} \dim \text{Im}(\Lambda_r|_{\{(x, s)\}} \otimes \mathcal{O}_S(-N)|_s \otimes L \rightarrow (E_{x, 1}/E_{x, i})) &= \dim \left(\frac{F|_{\{(x, s)\}}}{E_{x, i}} \right) \\ &= \text{rk} \left(\frac{F_{x, 1}}{E_{x, i} \cap F_{x, 1}} \right) = \text{rk} \left(\frac{F_{x, 1}}{F_{x, i}} \right) = \text{rk}(F_{x, 1}) - \text{rk}(F_{x, i}) = \text{rk}(F) - \text{rk}(F_{x, i}) \end{aligned}$$

Substituting the previous computation in condition (3.4.3) and taking into account (3.4.5) implies that condition (3.4.3) is equivalent to

$$\frac{\varepsilon_0 P_F(m) + M \operatorname{rk}(F) - \operatorname{wt}(F)}{\dim L} \geq \frac{\varepsilon_0 P_E(m) + M \operatorname{rk}(E) - \operatorname{wt}(E)}{P_E(N)} \quad (3.4.6)$$

Under the isomorphism $\alpha_E : H^0(X_s, E(N)) \rightarrow \mathbb{C}^{P(N)}$, L corresponds to a subspace of sections of $F(N)$. Therefore, $\dim L \leq \dim H^0(X_s, F(N)) = P_F(N) + h^1(X_s, F(N))$ and in order to prove that the previous condition holds it is sufficient to demonstrate that

$$\frac{\varepsilon_0 P_F(m) + M \operatorname{rk}(F) - \operatorname{wt}(F)}{P_F(N) + h^1(X_s, F(N))} \geq \frac{\varepsilon_0 P_E(m) + M \operatorname{rk}(E) - \operatorname{wt}(E)}{P_E(N)}$$

holds. Let us denote $\tau_F(N) = h^1(X_s, F(N)) / \operatorname{rk}(F)$. Applying Riemann-Roch theorem and dividing by the rank yields that the condition is equivalent to

$$\frac{\varepsilon_0(m+1-g) + M + \varepsilon_0\mu(F) - \eta(F)}{N+1-g+\mu(F) + \tau_F(N)} \geq \frac{\varepsilon_0(m+1-g) + M + \varepsilon_0\mu(E) - \eta(E)}{N+1-g+\mu(E)}$$

Subtracting ε_0 to both sides of the inequality gives

$$\frac{\varepsilon_0(m-N) + M - \eta(F) - \varepsilon_0\tau_F(N)}{N+1-g+\mu(F) + \tau_F(N)} \geq \frac{\varepsilon_0(m-N) + M - \eta(E)}{N+1-g+\mu(E)}$$

By the choice of N , both denominators are positive, so by multiplying and grouping one obtains

$$\begin{aligned} \mu(F) (\varepsilon_0(m-N) + M - \eta(E)) + \eta(F) (N+1-g+\eta(E)) &\leq \mu(E) (\varepsilon_0(m-N) + M) \\ &+ \eta(E) (N+1-g) - \tau_F(N) (\varepsilon_0(N+1-g+\mu(E)) + \varepsilon_0(m-N) + M - \eta(E)) \end{aligned}$$

Adding and subtracting $\mu(E)\eta(E)$ to the right hand side of the inequality yields

$$\begin{aligned} \mu(F) (\varepsilon_0(m-N) + M - \eta(E)) + \eta(F) (N+1-g+\mu(E)) \\ \leq \mu(E) (\varepsilon_0(m-N) + M - \eta(E)) + \eta(E) (N+1-g+\mu(E)) \\ - \tau_F(N) (\varepsilon_0(N+1-g+\mu(E)) + \varepsilon_0(m-N) + M - \eta(E)) \end{aligned}$$

Again, by the choice of ε_0 , $\varepsilon_0(m-N) + M - \eta(E) = N+1-g+\mu(E) > 0$ so we have to prove that for big enough m

$$\mu(F) + \eta(F) + (\varepsilon_0 + 1)\tau_F(N) \leq \mu(E) + \eta(E)$$

Adding $N+1-g$ to both sides of the inequality and applying Riemann-Roch theorem, this is equivalent to proving that

$$\begin{aligned} \frac{h^0(X_s, F(N))}{\operatorname{rk}(F)} + \eta(F) + \varepsilon_0\tau_F(N) &= \frac{P_F(N) + h^1(X_s, F(N))}{\operatorname{rk}(F)} + \eta(F) + \varepsilon_0\tau_F(N) \\ &= N+1-g+\mu(F) + \eta(F) + (\varepsilon_0+1)\tau_F(N) \leq N+1-g+\mu(E) + \eta(E) = \frac{P_E(N)}{\operatorname{rk}(E)} + \eta(E) \end{aligned}$$

As E is an element of $Q_5(s)$, it is a quotient $\mathbb{C}^{P(N)} \otimes \mathcal{O}_{X_s}(-N) \twoheadrightarrow E$. Let G be the image of $L \otimes \mathcal{O}_{X_s}(-N)$ under such quotient. By construction of Q_5 , it is clear that F is the image of G under the action of $\Lambda_{X_s,r}$

$$\begin{array}{ccc} \Lambda_{X_s,r} \otimes \mathcal{O}_{X_s}(-N) \otimes_{\mathbb{C}} L & \twoheadrightarrow & F \\ \downarrow & & \downarrow \\ \Lambda_{X_s,r} \otimes G & \twoheadrightarrow & E \end{array}$$

By Lemma 3.1.13, $F^{\text{sat}} = \text{Im}(\Lambda_{X_s,r} \otimes G \rightarrow E)^{\text{sat}}$ with the induced parabolic structure is a parabolic sub- Λ -module of E . Therefore, by Lemma 3.2.11

$$\frac{h^0(X_s, F(N))}{\text{rk}(F)} + \eta(F) \leq \frac{P_E(N)}{\text{rk}(E)} + \eta(E)$$

Let us distinguish two cases. If the inequality is an equality, then by Lemma 3.2.11 we know that $h^1(X_s, F(N)) = 0$, so $\tau_F(N) = 0$ and we are done. Otherwise, by Lemma 3.2.12, there exists a number $\delta > 0$ such that

$$\frac{h^0(X_s, F(N))}{\text{rk}(F)} + \eta(F) + \delta \leq \frac{P_E(N)}{\text{rk}(E)} + \eta(E) = \frac{h^0(X_s, E(N))}{\text{rk}(E)} + \eta(E)$$

As we have seen, the set of possible built subsheaves F is a bounded family over a projective curve. Therefore, by [Kle71, Theorem 3.13], the set of possible values for $\mu(F(N))$ is bounded from below. As we have

$$\begin{aligned} \tau_F(N) &= \frac{h^0(X_s, F(N))}{\text{rk}(F)} - (N + 1 - g) - \mu(F) \\ &\leq \frac{P_E(N)}{\text{rk}(E)} - (N + 1 - g) + \eta(E) - \eta(F) - \mu(F) \leq \frac{P_E(N)}{\text{rk}(E)} + \eta(E) - (N + 1 - g) - \min_F \{\mu(F)\} \end{aligned}$$

the set of possible values of $\tau_F(N)$ is bounded from above. Let τ be the maximum of such values. As N is fixed, there exists an $m_1 \geq m_0$ such that for $m \geq m_1$

$$\varepsilon_0 \tau_F(N) \leq \varepsilon_0 \tau < \delta$$

Therefore, for $m \geq m_1$, if $(E, E_{\bullet}, \varphi)$ is semistable then

$$\begin{aligned} \frac{h^0(X_s, F(N))}{\text{rk}(F)} + \eta(F) + \varepsilon_0 \tau_F(N) &< \frac{h^0(X_s, F(N))}{\text{rk}(F)} + \eta(F) + \delta \\ &\leq \frac{P_E(N)}{\text{rk}(E)} + \eta(E) \end{aligned}$$

Therefore, condition (3.4.6) holds for every $L \subseteq \mathbb{C}^{P(N)}$ and (E, E_{\bullet}) is GIT-semistable under embedding ψ_m for every $m \geq m_1$.

Suppose that $(E, E_{\bullet}, \varphi, \alpha)$ is strictly GIT semistable. Let $L \subseteq \mathbb{C}^{P(N)}$ such that we have an equality in (3.4.6). Take $F = \text{Im}(\Lambda_{X_s,r} \otimes \mathcal{O}_{X_s}(-N) \otimes_{\mathbb{C}} L \rightarrow E)$ as before. As $L \subseteq H^0(X_s, F(N))$, we may assume without loss of generality that $L = H^0(X_s, F(N))$. Then

$$\frac{h^0(X_s, F(N))}{\text{rk}(F)} + \eta(F) + \varepsilon_0 \tau_F(N) = \frac{P_E(N)}{\text{rk}(E)} + \eta(E)$$

If we had $\tau_F(N) \neq 0$ then, in particular,

$$\frac{h^0(X_s, F(N))}{\text{rk}(F)} + \eta(F) < \frac{P_E(N)}{\text{rk}(E)} + \eta(E)$$

So, using Lemma 3.2.12 as $0 < \epsilon_0 \tau_F(N) < \delta$

$$\begin{aligned} \frac{h^0(X_s, F(N))}{\text{rk}(F)} + \eta(F) + \epsilon_0 \tau_F(N) &< \frac{h^0(X_s, F(N))}{\text{rk}(F)} + \eta(F) + \delta \\ &\leq \frac{P_E(N)}{\text{rk}(E)} + \eta(E) \end{aligned}$$

contradicting the equality assumption in (3.4.6). Therefore, $\tau_F(N) = 0$ and we obtain

$$\frac{h^0(X_s, F(N))}{\text{rk}(F)} + \eta(F) = \frac{P_E(N)}{\text{rk}(E)} + \eta(E)$$

By Lemma 3.2.11, F must be saturated and, therefore, (F, F_\bullet, φ) is a parabolic sub- Λ -module with $\text{par-}\mu(F) = \text{par-}\mu(E)$, so (E, E_\bullet, φ) is strictly semistable. \square

Theorem 3.4.8. *Let $\mathcal{M}(\Lambda, P, \alpha, \bar{r}) : (\text{Sch}_S) \rightarrow (\text{Sets})$ denote the functor that associates each S -scheme $f : T \rightarrow S$ the set of isomorphism classes of semistable parabolic Λ -modules over $C \times T$ with Hilbert polynomial P and the given parabolic type modulo S -equivalence and tensoring by a line bundle over T . Let $\mathcal{M}^s(\Lambda, P, \alpha, \bar{r})$ be the subfunctor corresponding to isomorphism classes of stable parabolic Λ -modules. There exists a quasi-projective variety $M(\Lambda, P, \alpha, \bar{r})$ such that*

1. $M(\Lambda, P, \alpha, \bar{r})$ is a coarse moduli space for $\mathcal{M}(\Lambda, P, \alpha, \bar{r})$.
2. There is an open subscheme $M^s(\Lambda, P, \alpha, \bar{r}) \subseteq M(\Lambda, P, \alpha, \bar{r})$ which is a coarse moduli space for the functor $\mathcal{M}^s(\Lambda, P, \alpha, \bar{r})$. Moreover, it admits a locally universal family in the étale topology.

Proof. Let Θ be any very ample line bundle over R for which Lemma 3.4.7 holds. Then Lemma 3.4.7 implies that the subscheme of GIT-(semi-)stable points of R is R^s (respectively R^{ss}). Take the GIT quotient $M(\Lambda, P, \alpha, \bar{r}) := R^{ss} // \text{SL}_{P(N)}(\mathbb{C})$. By the discussion at the start of this section, $\mathcal{M}(\Lambda, P, \alpha, \bar{r})$ is the quotient of the functor \mathcal{R}^{ss} described in Theorem 3.3.4 by $\text{SL}_{P(N)}(\mathbb{C})$. By [Sim94, Proposition 1.11], $M(\Lambda, P, \alpha, \bar{r})$ is a universal categorical quotient of R^{ss} by $\text{SL}_{P(N)}(\mathbb{C})$, the projection map $R^{ss} \rightarrow M(\Lambda, P, \alpha, \bar{r})$ is affine and $M(\Lambda, P, \alpha, \bar{r})$ is quasi-projective. Moreover, $M^s(\Lambda, P, \alpha, \bar{r}) \subseteq M(\Lambda, P, \alpha, \bar{r})$ is an open sub-scheme whose preimage under the quotient morphism is R^s such that $R^s \rightarrow M^s(\Lambda, P, \alpha, \bar{r})$ is a universal geometric quotient. This proves that $M(\Lambda, P, \alpha, \bar{r})$ universally corepresents

$$\mathcal{R}^{ss} / \text{SL}_{P(N)}(\mathbb{C}) \cong \mathcal{M}(\Lambda, P, \alpha, \bar{r})$$

and $M^s(\Lambda, P, \alpha, \bar{r})$ universally corepresents

$$\mathcal{R}^s / \text{SL}_{P(N)}(\mathbb{C}) \cong \mathcal{M}^s(\Lambda, P, \alpha, \bar{r})$$

Therefore, in order to prove the Theorem it is enough to prove that the geometric points of $M(\Lambda, P, \alpha, \bar{r})$ coincide with the S-equivalence classes of parabolic Λ -modules. By [Sim94, Lemma 1.10], the closed points in $M(\Lambda, P, \alpha, \bar{r})$ are in one to one correspondence with the closed orbits of $\mathrm{SL}_{P(N)}(\mathbb{C})$ in R^{ss} . Therefore, it is enough to prove that the orbit of any semistable parabolic Λ -module (E, E_\bullet) contains its graduate $\mathrm{Gr}(E)$ by the Jordan-Hölder filtration and that the orbit of a graduate object $\mathrm{Gr}(E)$ is closed. The proof is then completely analogous to the one given for [Sim94, Theorem 1.21.3] (used also in [Sim94, Theorem 4.7.3] to prove the same property for non-parabolic Λ -modules). \square

3.5 Residual parabolic Λ -modules

One of the principal uses of enhancing a vector bundle with a parabolic structure is to control the behavior of some “field” such as a logarithmic connection or a Higgs field near a puncture in a smooth Riemann surface. This is usually done through the control of some kind of *residue* of the structure around the parabolic points of the surface.

Following the definitions of residue of a parabolic Higgs field or residue of a parabolic connection (parabolic \mathcal{D}_C -module) given by Simpson [Sim90], the residue at a parabolic point is an endomorphism of the fiber of the underlying bundle over the point induced by the action of the field or connection respectively.

While studying the moduli spaces of these geometric structures and the correspondences between them, some restrictions over the residues of both the Higgs field and the connection appear naturally. For example, in the case of parabolic Higgs bundles, for every stable parabolic vector bundle (E, E_\bullet) , the cotangent space at (E, E_\bullet) to the moduli space of parabolic vector bundles can be canonically identified through Serre duality with $H^0(\mathrm{SPEnd}(E, E_\bullet) \otimes K(D))$. Therefore, every element of the cotangent bundle corresponds to a stable parabolic Higgs bundle. All the parabolic Higgs fields φ obtained this way share the property of being strongly parabolic, i.e., for every $x \in D$ and for every $i = 1, \dots, l_x$

$$\mathrm{Res}(\varphi, x)(E_{x,i}) \subseteq E_{x,i+1} \quad (3.5.1)$$

In fact, if we restrict ourselves to studying strongly parabolic Higgs fields, then the cotangent bundle of the moduli space of stable parabolic vector bundles fits as an open dense subset of the moduli space of stable strongly parabolic Higgs bundles. We can also understand the strongly parabolic condition as a control of the eigenvalues of the Higgs field at the parabolic points. In this sense, it is equivalent to imposing that the residue has a null spectrum.

On the other hand, through Simpson’s correspondence we know that there exists an equivalence of categories between stable parabolic Higgs bundles and stable filtered \mathcal{D}_C -modules [Sim90]. If (E, E_\bullet, ∇) is a parabolic connection corresponding to an stable strongly parabolic Higgs bundle, then the action of the residue $\mathrm{Res}(\nabla, x)$ at the fiber $E|_x$ must satisfy the following two conditions

1. The eigenvalues of $\mathrm{Res}(\nabla, x)$ must coincide with the parabolic weights and,
2. $\mathrm{Res}(\nabla, x)$ must act on $E_{x,i}/E_{x,i+1}$ as multiplication by α_i .

These conditions can be reformulated as imposing that for every $x \in D$ and every i

$$(\text{Res}(\nabla, x) - \alpha_{x,i} \text{Id}_{E|_x})(E_{x,i}) \subseteq E_{x,i+1} \quad (3.5.2)$$

As another example, in order to define parabolic λ -connections we must impose the following condition on the residue, serving as a sort of interpolation between the other two

$$(\text{Res}(\nabla, x) - \lambda \alpha_{x,i} \text{Id}_{E|_x})(E_{x,i}) \subseteq E_{x,i+1}$$

This will be explored in more detail in the last part of this chapter (section 3.7).

Through this section we aim to give a suitable definition for the residue of a parabolic Λ -module that generalizes the ones given by Simpson [Sim90] in the previous scenarios. Using this definition, we will present a “*residual*” condition on parabolic Λ -modules that unifies the previous examples and we will show that the moduli of semistable residual parabolic Λ -modules exists as a closed subscheme of the one constructed in the previous section.

Let Λ be a sheaf of rings of differential operators such that $\Lambda|_{\overline{D}}$ is locally free. Let (E, E_\bullet, φ) be a parabolic Λ -module. By definition, for every parabolic point $x \in D$, the image of $\Lambda \otimes E(-\{x\} \times S)$ by the morphism $\varphi : \Lambda \otimes E \rightarrow E$ lies in $E(-\{x\} \times S)$. Therefore, if $i_x : S \cong \{x\} \times S \hookrightarrow C \times S$ is the canonical inclusion we have a commutative diagram of sheaves of $(\mathcal{O}_X, \mathcal{O}_X)$ -modules

$$\begin{array}{ccccccc} \Lambda \otimes_{\mathcal{O}_X} E(-\{x\} \times S) & \longrightarrow & \Lambda \otimes_{\mathcal{O}_X} E & \longrightarrow & \Lambda \otimes_{\mathcal{O}_X} (i_x)_* E|_{\{x\} \times S} & \longrightarrow & 0 \\ \downarrow & & \downarrow \varphi & & \downarrow & & \\ 0 \longrightarrow & E(-\{x\} \times S) & \longrightarrow & E & \xrightarrow{\text{ev}} & (i_x)_* E|_{\{x\} \times S} & \longrightarrow 0 \end{array}$$

and we obtain a morphism

$$\varphi|_{\{x\} \times D} : \Lambda \otimes_{\mathcal{O}_X} (i_x)_* E|_{\{x\} \times S} \longrightarrow (i_x)_* E|_{\{x\} \times S}$$

Taking the pullback by $i_x : S \hookrightarrow C \times S$ we obtain an induced morphism

$$\text{Res}(\varphi, x) : \Lambda|_{\{x\} \times S} \otimes_{\mathcal{O}_S} E|_{\{x\} \times S} \rightarrow E|_{\{x\} \times S}$$

Definition 3.5.1 (Total residue of a Λ -module). *We call*

$$\text{Res}(\varphi, x) : \Lambda|_{\{x\} \times S} \otimes_{\mathcal{O}_S} E|_{\{x\} \times S} \longrightarrow E|_{\{x\} \times S}$$

the total residue of the parabolic Λ -module (E, E_\bullet, φ) at the point $x \in D$.

As $\Lambda \otimes E \rightarrow E$ preserves the parabolic structure, then for every $x \in D$, $\text{Res}(\varphi, x)$ preserves the parabolic filtration in the sense that for every $i = 1, \dots, l_x$

$$\text{Res}(\varphi, x) (\Lambda|_{\{x\} \times S} \otimes_{\mathcal{O}_S} E_{x,i}) \subseteq E_{x,i}$$

As an explicit example of this construction, let S be a point and let us consider a parabolic connection $\nabla : E \rightarrow E \otimes K_C(D)$. Let $x \in D$ be a parabolic point and let z be a local coordinate in a neighborhood of x . Then there is a matrix A_x such that ∇ can be locally written as

$$\nabla(v) = A_x v \frac{dz}{z} + dv$$

where v is a local section of E around x . Now let us consider the operator $\nabla' : T_C(-D) \otimes E \rightarrow E$ obtained from ∇ by contraction. It corresponds to the action of $\Lambda_1^{\text{DR}, \log D}$ on E . If \mathcal{X} is a local section of $T_C(-D)$ and v is a local section of E around $x \in D$, then

$$\nabla'(\mathcal{X}, v) = A_x v \mathcal{X}(v)/z + \mathcal{X}(v)$$

In particular, as \mathcal{X} is locally written as $\mathcal{X} = tz \frac{\partial}{\partial z}$ for some local regular function t , then

$$\nabla'(\mathcal{X}, v) = \nabla' \left(tz \frac{\partial}{\partial z}, v \right) = tA_x v + tz \frac{\partial v}{\partial z}$$

Observe that the second summand of the right hand side is always a local section of $E(-x)$. Moreover, if $v \in E(-x)$ then the first factor also belongs to $E(-x)$, so ∇' sends $T_C(-D) \otimes E(-D)$ to $E(-D)$. As the evaluation of the second summand at $z = 0$ is always 0, the evaluation

$$\nabla'(\mathcal{X}, v)|_{z=0} = (tA_x v)|_{z=0} = t(0)A_x v(0)$$

only depends on $\mathcal{X}|_{z=0}$ and $v|_{z=0}$, so we obtain a morphism

$$\begin{aligned} \text{Res}(\nabla', x)|_{T_C(-D)|_x} : T_C(-D)|_x \otimes E|_x &\cong \mathcal{O}_x \otimes E|_x \longrightarrow E|_x \\ tz \frac{\partial}{\partial z}|_{z=0} \otimes v &\longmapsto t \otimes v \longmapsto tA_x v \end{aligned}$$

Notice that through the canonical isomorphism $\mathcal{O}_x \otimes E|_x \cong E|_x$, the morphism $\text{Res}(\nabla', x)|_{T_C(-D)|_x} \in \text{End}(E_x)$ coincides with the usual notion of residue of ∇ , in the sense of the order -1 coefficient in the Laurent series of ∇ around the point x .

Now let us consider the complete action of $\Lambda_1^{\text{DR}, \log D} = \mathcal{O}_C \oplus T_C(-D)$ on E . It is locally given by

$$\nabla'(f + \mathcal{X}, v) = fv + \nabla'(\mathcal{X}, v)$$

Where f is a locally regular function around $x \in D$. Once more the restriction of $\nabla'(f + \mathcal{X}, v)$ to $z = 0$ only depends on $f(0)$, $\mathcal{X}|_{z=0}$ and $v|_{z=0}$, so we obtain a morphism

$$\begin{aligned} \text{Res}(\nabla', x)|_{\Lambda_1^{\text{DR}, \log D}|_x} : \Lambda_1^{\text{DR}, \log D}|_x \otimes E|_x &\cong (\mathcal{O}_x \oplus \mathcal{O}_x) \otimes E|_x \longrightarrow E|_x \\ (f + tz \frac{\partial}{\partial z}|_{z=0}) \otimes v &\longmapsto (f, t) \otimes v \longmapsto fv + tA_x v \end{aligned}$$

Similarly, if $\varphi : E \rightarrow E \otimes K_C(D)$ is a Higgs field, then around $x \in D$ it is locally given as

$$\varphi(v) = A_x v \frac{dz}{z}$$

for some matrix A_x . If we consider the induced morphism $\varphi' : (\mathcal{O}_C \oplus T_C(D)) \otimes E \rightarrow E$ given by

$$\varphi'(f + \mathcal{X}, v) = fv + A_x v \mathcal{X}(z)/z$$

then the evaluation of this expression at $z = 0$ clearly only depends on $f(0)$, $\mathcal{X}|_{z=0}$ and $v|_{z=0}$ so it induces a morphism

$$\begin{aligned} \text{Res}(\varphi', x)|_{\Lambda_1^{\text{Higgs}, \log D}|_x} : \Lambda_1^{\text{Higgs}, \log D}|_x \otimes E|_x &\cong (\mathcal{O}_x \oplus \mathcal{O}_x) \otimes E|_x \longrightarrow E|_x \\ (f + tz \frac{\partial}{\partial z}|_{z=0}) \otimes v &\longmapsto (f, t) \otimes v \longmapsto fv + tA_x v \end{aligned}$$

Observe that if (E, E_\bullet, φ) is a parabolic Λ -module and $R \in H^0(S, \Lambda|_{\{x\} \times S})$, then composing R with $\text{Res}(\varphi, x)$ yields an endomorphism of the fiber $E|_{\{x\} \times S}$ that preserves the parabolic filtration.

$$\text{Res}_R(\varphi, x) : E|_{\{x\} \times S} = \mathcal{O}_X|_{\{x\} \times S} \otimes E|_{\{x\} \times S} \xrightarrow{f^* R} f^* \Lambda|_{\{x\} \times S} \otimes E|_{\{x\} \times S} \xrightarrow{\text{Res}(\varphi, x)} E|_{\{x\} \times S}$$

Definition 3.5.2. Let Λ be a sheaf of rings of differential operators over $C \times S$ flat over S such that $\Lambda|_{\overline{D}}$ is locally free. A residual condition for Λ over D is a set of sections $\overline{R} = \{R_{x,i}\}$ with $R_{x,i} \in H^0(S, \Lambda|_{\{x\} \times S})$ for each $x \in D$ and each $i = 1, \dots, l_x$. Given a residual condition \overline{R} , for each $f : T \rightarrow S$ and each family of parabolic Λ -modules over T , (E, E_\bullet, φ) we obtain morphisms

$$\text{Res}_{R_{x,i}}(\varphi, x) : E|_{\{x\} \times T} = \mathcal{O}_X|_{\{x\} \times T} \otimes E|_{\{x\} \times T} \xrightarrow{f^*(R_{x,i})} f^* \Lambda|_{\{x\} \times T} \otimes E|_{\{x\} \times T} \xrightarrow{\text{Res}(\varphi, x)} E|_{\{x\} \times T}$$

So we obtain an endomorphism $\text{Res}_{R_{x,i}}(\varphi, x) \in H^0(T, \text{End}_{\mathcal{O}_T}(E|_{\{x\} \times T}))$. Moreover, as Λ preserves the parabolic filtration, $\text{Res}_{R_{x,i}}(\varphi, x)$ preserves the parabolic filtration over x for every $i = 1, \dots, l_x$. If \overline{R} is a residual condition for Λ over D , we say that a parabolic Λ -module (E, E_\bullet) is \overline{R} -residual if for every $x \in D$ and every $i = 1, \dots, l_x$

$$\text{Res}_{R_{x,i}}(\varphi, x)(E_{x,i}) \subseteq E_{x,i+1}$$

For example, if we take

$$R_{x,i}^{\text{DR}} = \left(-\lambda_{x,i}, z \frac{\partial}{\partial z} \Big|_{z=0} \right) \in H^0(x, \Lambda_1^{\text{DR}, \log D}|_x) \subsetneq H^0(x, \Lambda^{\text{DR}, \log D}|_x)$$

Then for the connection $\nabla : E \rightarrow E \otimes K(D)$ in the previous example

$$\text{Res}_{R_{x,i}^{\text{DR}}}(\nabla, x) = A_x - \lambda_i \text{Id} \in \text{End}(E|_x)$$

Therefore a connection is \overline{R}^{DR} -residual if

1. The eigenvalues of the residue at each parabolic point $x \in D$ are $\{\lambda_{x,i}\}$ respectively and
2. the residue acts on $E_{x,i}/E_{x,i+1}$ by multiplication by $\lambda_{x,i}$

Therefore, we can control the eigenvalues of a connection through an \overline{R}^{DR} -residual condition. For example, taking $\lambda_{x,i} = \alpha_{x,i}$ we recover the residue condition of a parabolic connection described at the start of the section. This can be also achieved in the context of Higgs bundles. For example, if we take

$$R_{x,i}^{\text{Higgs}} = z \frac{\partial}{\partial z} \Big|_{z=0} \in H^0(x, \Lambda_1^{\text{Higgs}, \log D}|_x) \subsetneq H^0(x, \Lambda^{\text{Higgs}, \log D}|_x)$$

Then $\text{Res}_{R_{x,i}^{\text{Higgs}}}(\varphi, x) = \varphi|_x$ so a Higgs bundle $\varphi : E \rightarrow E \otimes K(D)$ is $\overline{R}^{\text{Higgs}}$ -residual if and only if it is strongly parabolic.

Theorem 3.5.3. *Let Λ be a sheaf of rings of differential operators over $C \times S$ flat over S such that $\Lambda|_{\overline{D}}$ is locally free. Let \overline{R} , be a residual condition for Λ over D . Let $\mathcal{M}(\Lambda, \overline{R}, P, \alpha, \bar{r}) : (\text{Sch}_S) \rightarrow (\text{Sets})$ denote the functor that associates each S -scheme $f : T \rightarrow S$ the set of isomorphism classes of semistable \overline{R} -residual parabolic Λ -modules over $C \times T$ with Hilbert polynomial P and the given parabolic type modulo S -equivalence and tensoring by a line bundle over T . Let $\mathcal{M}^s(\Lambda, \overline{R}, P, \alpha, \bar{r})$ be the subfunctor corresponding to classes of stable parabolic Λ -modules. There exists a quasi-projective variety $M(\Lambda, \overline{R}, P, \alpha, \bar{r})$ such that*

1. $M(\Lambda, \overline{R}, P, \alpha, \bar{r})$ is a coarse moduli space for the functor $\mathcal{M}(\Lambda, \overline{R}, P, \alpha, \bar{r}) \subset \mathcal{M}(\Lambda, P, \alpha, \bar{r})$.
2. There is an open subscheme $M^s(\Lambda, \overline{R}, P, \alpha, \bar{r}) \subseteq M(\Lambda, \overline{R}, P, \alpha, \bar{r})$ which is a coarse moduli space for the functor $\mathcal{M}^s(\Lambda, \overline{R}, P, \alpha, \bar{r})$. Moreover, it admits a locally universal family in the étale topology.

Proof. Let R be the quasi-projective scheme described by Theorem 3.3.4. Let $(\mathcal{E}, \mathcal{E}_\bullet)$ be the Λ -module underlying the corresponding universal object and $\varphi^{\text{univ}} : \pi^* \Lambda \otimes \mathcal{E} \rightarrow \mathcal{E}$ be the universal action of Λ on \mathcal{E} . Let $f : T \rightarrow S$. A family of rigidified parabolic Λ -modules $(E, E_\bullet, \varphi, \alpha)$ corresponding to a T point $e : T \rightarrow R$ is \overline{R} -residual if for every $x \in D$ and $1 \leq i \leq l_x$ the following composition of morphisms

$$e^* \mathcal{E}_{x,i} \hookrightarrow e^* \mathcal{E}|_{x,1} \xrightarrow{f^*(R_{x,i})} f^*(\Lambda|_{\{x\} \times S}) \otimes e^* \mathcal{E}_{x,1} \xrightarrow{e^* \text{Res}(\varphi^{\text{univ}}, x)} e^* \mathcal{E}_{x,1} \twoheadrightarrow e^*(\mathcal{E}_{x,1}/\mathcal{E}_{x,i+1})$$

is zero. By [Yok93, Lemma 4.3], there is a closed subscheme $R_{\text{Res}} \subseteq R$ parameterizing \overline{R} -residual Λ -modules of R . Let $R_{\text{Res}}^{ss} := R^{ss} \cap R_{\text{Res}}$ and $R_{\text{Res}}^s := R^s \cap R_{\text{Res}}$. It is clear that R_{Res} is invariant under the action of $\text{SL}_{P(N)}(\mathbb{C})$ on R . Therefore, the quotient of R^{ss} by $\text{SL}_{P(N)}(\mathbb{C})$ restricts to a quotient of R_{Res}^{ss} .

Theorem 3.4.8 applies and we get that the closed subscheme

$$M(\Lambda, \overline{R}, P, \alpha, \bar{r}) = R_{\text{Res}}^{ss} // \text{SL}_{P(N)}(\mathbb{C}) \hookrightarrow R^{ss} // \text{SL}_{P(N)}(\mathbb{C}) = M(\Lambda, P, \alpha, \bar{r})$$

is a coarse quasi-projective moduli space for the given moduli functor $\mathcal{M}(\Lambda, \overline{R}, P, \alpha, \bar{r})$. Similarly $M^s(\Lambda, \overline{R}, P, \alpha, \bar{r}) \subseteq M^s(\Lambda, P, \alpha, \bar{r})$ is a quasi-projective coarse moduli space for $\mathcal{M}^s(\Lambda, \overline{R}, P, \alpha, \bar{r})$.

The existence of a local universal family in the étale topology on $M^s(\Lambda, \overline{R}, P, \alpha, \bar{r})$ is obtained by restriction of local universal families over $M^s(\Lambda, P, \alpha, \bar{r})$. \square

3.6 Existence of a universal family

In this section we prove that, under certain generic hypothesis, the moduli spaces of stable parabolic Λ -modules previously constructed are fine moduli spaces, i.e., there exists a universal family and the corresponding scheme represents the moduli functor. This will be achieved generalizing the proof in [BY99, Section 3].

Lemma 3.6.1. *Let (E, E_\bullet) be a parabolic vector bundle on (C, D) such that the underlying vector bundle E is of type e , degree d and rank r . Let (H, H_\bullet) be a parabolic line bundle of degree h . Let $M = |D|$. If*

$$d > rh + r(2g - 2 + M) + r(r - 1)e$$

then $h^1(\mathrm{PHom}((H, H_\bullet), (E, E_\bullet))) = 0$.

Proof. By parabolic Serre duality we obtain

$$\begin{aligned} h^1(\mathrm{PHom}((H, H_\bullet), (E, E_\bullet))) &= h^0(\mathrm{SPHom}((E, E_\bullet), (H, H_\bullet)) \otimes K(D)) \\ &\leq h^0(\mathrm{PHom}((E, E_\bullet), (H, H_\bullet)) \otimes K(D)) \end{aligned}$$

Where PHom and SPHom denote the sheaves of parabolic morphisms and strongly parabolic morphisms respectively.

Let $\varphi : (E, E_\bullet) \rightarrow (H, H_\bullet) \otimes K(D)$ be a nonzero parabolic morphism and let (K, K_\bullet) be the kernel of the morphism φ endowed with the induced parabolic structure from (E, E_\bullet) . Then yields

$$\deg(K) \geq \deg(E) - \deg(H \otimes K(D)) = d - h - (2g - 2 + M)$$

On the other hand, as (E, E_\bullet) is of type e we have

$$\mu(K) = \frac{\deg(K)}{r-1} \leq \mu(E) + e = \frac{d}{r} + e$$

Solving for $\deg(K)$ in the second inequality and substituting in the first one yields

$$(r-1)d + r(r-1)e \geq r \deg(K) \geq rd - rh - r(2g - 2 + M)$$

Therefore

$$d \leq rh + r(2g - 2 + M) + r(r-1)e$$

□

Now let $(\mathcal{E}, \mathcal{E}_\bullet, \Phi, A)$ be the universal rigidified Λ -module over R^s defined in Theorem 3.3.4. It is clear that the action of $\mathrm{SL}_{P(N)}(\mathbb{C})$ on $(\mathcal{E}, \mathcal{E}_\bullet, \Phi)$ is trivial, but the center of $\mathrm{GL}_{P(N)}(\mathbb{C})$ acts on $(\mathcal{E}, \mathcal{E}_\bullet, \Phi)$ by dilations on the fibers. Observe that the action of \mathbb{C}^* on the action Φ is also trivial. Therefore, if there exists a line bundle L over R^s with a natural lift of the $\mathrm{GL}_{P(N)}(\mathbb{C})$ action such that the center \mathbb{C}^* acts by multiplication on L , then the rigidified Λ -module

$$(\mathcal{E} \otimes \pi_{R^s}^* L^{-1}, \mathcal{E}_\bullet \otimes \pi_{R^s}^* L^{-1}, \Phi \otimes \mathrm{Id}, \alpha \otimes \mathrm{Id})$$

is $\mathrm{GL}_{P(N)}(\mathbb{C})$ -equivariant and, therefore, it projects to a universal family of stable parabolic Λ -modules over $\mathcal{M}^s(\Lambda, P, \alpha, \bar{r})$.

Lemma 3.6.2. *Given a parabolic type $\bar{r} = \{r_{x,i}\}$, let*

$$m_{x,i} = r_{x,i+1} - r_{x,i}$$

for $i = 1, \dots, l_x$. If the great common divisor of the numbers $\{d, m_{x,i} | x \in D, 1 < i \leq l_x\}$ is one, then there exists a line bundle L over R^s with natural action of $\mathrm{GL}_{P(N)}(\mathbb{C})$ such that the elements $\gamma \cdot \mathrm{Id} \in \mathrm{GL}_{P(N)}(\mathbb{C})$ act by multiplication by γ .

Proof. From Lemma 3.2.4, we know that there is some $e \in \mathbb{R}$ such that every stable parabolic Λ -module underlying a point in R^s is of type e . Take a line bundle H with

$$\deg(H) = h < \frac{d}{r} - 2g + 2 - M - (r-1)e$$

For every choice $\kappa : D \rightarrow \mathbb{Z}$ such that $1 \leq \kappa(x) \leq l_x + 1$, set

$$\beta_x (= \beta_{x,1}) = \begin{cases} \alpha_{x,\kappa(x)} & \text{if } \kappa(x) \leq l_x \\ \frac{1+\alpha_{x,l_x}}{2} & \text{if } \kappa(x) = l_x + 1 \end{cases}$$

and endow H with the trivial parabolic structure for the system of weights β . Let us denote it by (H, H_\bullet^κ) . Moreover, let

$$\chi(\kappa, h) = d + r(1 - g - h) - \sum_{x \in D} \sum_{i=1}^{\kappa(x)-1} m_{x,i}$$

By the previous lemma, for every stable parabolic Λ -module (E, E_\bullet, φ) we have $h^1(\text{PHom}((H, H_\bullet^\kappa), (E, E_\bullet))) = 0$. Applying Riemann-Roch theorem yields

$$h^0(\text{PHom}((H, H_\bullet^\kappa), (E, E_\bullet))) = \chi(\kappa, h)$$

so $\mathcal{E}' = H^0(C \times R^s/R^s, \text{PHom}(\pi_C^*(H, H_\bullet^\kappa), (\mathcal{E}, \mathcal{E}_\bullet)))$ is a locally free sheaf of rank $\chi(\kappa, h)$ over R^s . Let $L(\kappa, h)$ be its determinant. By construction $\text{GL}_{P(N)}(\mathbb{C})$ acts on \mathcal{E}' and there is an induced action on $L(\kappa, h)$ such that $\gamma \cdot \text{Id}$ acts as multiplication by $\gamma^{\chi(\kappa, h)}$.

Given $a_1, \dots, a_M \in \mathbb{Z}$, $\kappa_1, \dots, \kappa_M : D \rightarrow \mathbb{Z}$ and $h_1, \dots, h_M \in \mathbb{Z}$, consider the bundle

$$\bigotimes_{i=1}^M L(\kappa_i, h_i)^{a_i}$$

with the induced $\text{GL}_{P(N)}(\mathbb{C})$ action. Then $\gamma \cdot \text{Id}$ acts as multiplication by $\gamma^{\sum_{i=1}^M a_i \chi(\kappa_i, h_i)}$. Therefore it is enough to prove that there exist a_i, κ_i and h_i such that $\sum_{i=1}^M a_i \chi(\kappa_i, h_i) = 1$. Let $\kappa_{x,i}, \kappa_{x,i}^+ : D \rightarrow \mathbb{Z}$ be given by

$$\begin{aligned} \kappa_{x,i}(y) &= \begin{cases} i-1 & x=y \\ 0 & x \neq y \end{cases} \\ \kappa_{x,i}^+(y) &= \begin{cases} i & x=y \\ 0 & x \neq y \end{cases} \end{aligned}$$

Then for every h yields

$$\begin{aligned} \chi(\kappa_{x,i}^+, h) - \chi(\kappa_{x,i}, h) &= m_{x,i} \\ \chi(\kappa, h) - \chi(\kappa, h-1) &= r \\ \chi(0, h) - (1-g-h)(\chi(0, h) - \chi(0, h-1)) &= d \end{aligned}$$

As $\text{GCD}(\{m_{x,i}, r, d\}) = \text{GCD}(\{m_{x,i}, d\}) = 1$ the lemma follows. \square

The previous discussion leads to the following theorem.

Theorem 3.6.3. *For every $x \in D$ and $i = 1, \dots, l_x$ let $m_{x,i} = r_{x,i+1} - r_{x,i} = \dim(E_{x,i}) - \dim(E_{x,i+1})$. Let d be the degree of the underlying vector bundle, so that $P(m) = r(m+1-g) + d$. If the great common divisor of the numbers $\{d, m_{x,i} | x \in D, 1 < i \leq l_x\}$ is one then*

1. *The moduli space of stable parabolic Λ -modules is a fine moduli space, i.e., there exists a universal family $(\mathcal{E}, \mathcal{E}_\bullet, \Phi)$ over $\mathcal{M}^s(\Lambda, P, \alpha, \bar{r})$.*
2. *For every residual condition \bar{R} , the moduli space of stable residual parabolic Λ -modules is a fine moduli space, i.e., there exists a universal family $(\mathcal{E}, \mathcal{E}_\bullet, \Phi)$ over $\mathcal{M}^s(\Lambda, \bar{R}, P, \alpha, \bar{r})$.*

Proof. If the coprimality condition holds then the discussion at the start of the section combined with Lemma 3.6.2 proves that $\mathcal{M}^s(\Lambda, P, \alpha, \bar{r})$ admits a universal parabolic Λ -module. Restricting it to the closed subscheme $\mathcal{M}^s(\Lambda, \bar{R}, P, \alpha, \bar{r}) \subseteq \mathcal{M}^s(\Lambda, \bar{R}, P, \alpha, \bar{r})$ we obtain the second desired universal family. \square

Observe that, in particular, if take Λ to be trivial then we recover precisely the numerical condition on \bar{r} given by Boden and Yokogawa [BY99, Proposition 3.2] for the existence of a universal family on the moduli space of parabolic vector bundles. Moreover, if the parabolic type is trivial and $\alpha = 0$ the moduli space coincides with the moduli space of stable Λ -modules and the numerical condition reduces to asking for the rank and degree to be coprime. For trivial Λ or for $\Lambda = \Lambda^{\text{Higgs}}$ this is known to be a necessary and sufficient condition [Ram73].

Notice that, while for Λ -modules on the compact case this coprimality condition implies that there does not exist any strictly semistable object, for nontrivial parabolic structures there exist non-generic systems of weights such that there exist strictly semistable Λ -modules in $M(\Lambda, P, \alpha, \bar{r})$ and simultaneously the subscheme $\mathcal{M}^s(\Lambda, P, \alpha, \bar{r}) \subsetneq M(\Lambda, P, \alpha, \bar{r})$ admits a universal family. In particular, we have

Corollary 3.6.4. *If \bar{r} is a full flag parabolic type then*

1. *$\mathcal{M}^s(\Lambda, P, \alpha, \bar{r})$ is a fine moduli space.*
2. *$\mathcal{M}^s(\Lambda, \bar{R}, P, \alpha, \bar{r})$ is a fine moduli space.*

3.7 Moduli space of parabolic λ -connections

Let ξ be a line bundle over C , let α be a fixed system of weights over D and $\bar{r} = \{r_{x,i}\}$ a parabolic type. Let us suppose that $\deg(\xi) = -\sum_{x \in D} \sum_{i=1}^{l_x} \alpha_{x,i}$. Fixing a line bundle and a system of weights α over C allows us to describe canonically a parabolic line bundle over C , (ξ, ξ_β) , taking the underlying vector bundle as ξ and defining trivial filtrations over each $x \in D$ with parabolic weight

$$\beta_x := \beta_{x,1} = \sum_{i=1}^{l_x} \alpha_{x,i} (r_{x,i+1} - r_{x,i})$$

As ξ has rank one, any parabolic structure on ξ consists of trivial filtrations. It is possible that for some $x \in D$, $\beta_x \geq 1$. Taking into account the definition for the

parabolic structure in terms of left continuous filtrations given by Simpson [Sim90], a parabolic line bundle ξ with jumps at weights β_x for each $x \in D$ such that $\xi_{\beta_x, x} = \xi_x$ is the same as a trivial filtration for the bundle

$$\xi \left(\sum_{x \in D} \lfloor \beta_x \rfloor x \right) \quad (3.7.1)$$

with parabolic weights $\{\beta_x - \lfloor \beta_x \rfloor\}_{x \in D}$.

Thus, the value of the jump β_x completely defines the parabolic structure on ξ . By construction, we get that

$$\text{pardeg}(\xi) = \deg(\xi) + \sum_{x \in D} \beta_x = \deg(\xi) + \sum_{x \in D} \sum_{i=1}^{l_x} \alpha_{x,i} = 0$$

The line bundle ξ can be given the structure of a parabolic Higgs bundle canonically taking a zero Higgs field. In fact, as the rank of ξ is one, every traceless Higgs field over ξ must be zero, so $\mathcal{M}_{\text{Higgs}}(1, \beta, \xi)$ consists exactly of the point $(\xi, \xi_\beta, 0)$.

Let (E, E_\bullet, Φ) be a traceless strongly parabolic $\text{SL}_r(\mathbb{C})$ -Higgs bundle with parabolic system of weights α such that $\det(E) = \xi$. Taking the r -th exterior power, the morphism Φ induces a morphism $\bigwedge^r E \rightarrow \bigwedge^r E \otimes K(D)$ locally given by the trace of Φ . As $\text{tr}(\Phi) = 0$, the induced morphism is the zero morphism.

Thus, taking the determinant, every parabolic Higgs bundles $[(E, E_\bullet, \Phi)] \in \mathcal{M}_{\text{Higgs}}(r, \alpha, \xi)$ induces the same parabolic Higgs bundle $(\xi, \xi_\beta, 0)$.

Using the Simpson correspondence [Sim90] between parabolic Higgs bundles of parabolic degree 0 and parabolic connections of parabolic degree 0, the parabolic Higgs bundle $(\xi, \xi_\beta, 0)$ corresponds to a parabolic connection $(\xi, \xi_\beta, \nabla_{\xi, \beta})$ with the same parabolic weights β , such that $\text{Res}(\nabla_{\xi, \beta}, x) = \beta_x \text{Id}$ for every $x \in D$.

Let (E', E'_\bullet, ∇) be the parabolic connection corresponding to the Higgs bundle (E, E_\bullet, Φ) under the Simpson correspondence. Taking the r -th exterior power, ∇ induces a morphism

$$\tilde{\nabla} : \bigwedge^r E \rightarrow \bigwedge^r E \otimes K(D) \quad .$$

As the Simpson correspondence is an equivalence of categories preserving the exterior product [Sim90, Theorem 2], the wedge product of (E', E'_\bullet, ∇) must be the image of the wedge product of (E, E_\bullet, Φ) . Therefore, the morphism $\tilde{\nabla}$ must coincide with $\nabla_{\xi, \beta}$. This leads up to the following definition of parabolic λ -connection for the group $\text{SL}_r(\mathbb{C})$.

Definition 3.7.1. *For a fixed line bundle ξ , a system of weights α and a given $\lambda \in \mathbb{C}$ a parabolic λ -connection on C (for the group $\text{SL}_r(\mathbb{C})$) is a quadruple $(E, E_\bullet, \nabla, \lambda)$ where*

1. λ is a complex number.
2. $(E, E_\bullet) \rightarrow C$ is a parabolic vector bundle of rank r and weight system α together with an isomorphism $\bigwedge^r E \cong \xi$.
3. $\nabla : E \rightarrow E \otimes K(D)$ is a \mathbb{C} -linear homomorphism of sheaves over the underlying vector space of E satisfying the following conditions

- (a) If f is a locally defined holomorphic function on C and s is a locally defined holomorphic section of E then

$$\nabla(fs) = f \cdot \nabla(s) + \lambda \cdot s \otimes df$$

- (b) For each $x \in D$ the homomorphism induced in the filtration over the fiber E_x satisfies

$$\nabla(E_{x,i}) \subseteq E_{x,i} \otimes K(D)|_x$$

- (c) For every $x \in D$ and every $i = 1, \dots, l_x$ the action of $\text{Res}(\nabla, x)$ on $E_{x,i}/E_{x,i+1}$ is the multiplication by $\lambda\alpha_{x,i}$. Since $\text{Res}(\nabla, x)$ preserves the filtration, it acts on each quotient.
- (d) The operator $\bigwedge^r E \longrightarrow (\bigwedge^r E) \otimes K(D)$ induced by ∇ coincides with $\lambda \cdot \nabla_{\xi, \beta}$.

We also have the following natural notion of stability for λ -connections.

Definition 3.7.2. A parabolic λ -connection (E, E_\bullet, ∇) is (semi-)stable if and only if for every parabolic subsheaf $(F, F_\bullet) \subseteq (E, E_\bullet)$ preserved by ∇

$$\text{par-}\mu(F)(\leq) < \text{par-}\mu(E)$$

Given a parabolic λ -connection $(E, E_\bullet, \nabla, \lambda)$, the connection induces a parabolic morphism $\nabla' : (K(D))^\vee \otimes E \rightarrow E$ that satisfies that for every local section v of $(K(D))^\vee$, each locally defined holomorphic function on C and each local section f of E

$$\nabla'(v \otimes (fs)) = f\nabla'(v \otimes s) + \lambda df(v) \cdot s \quad (3.7.2)$$

Then we get a morphism $(\text{Id}|_E \oplus \nabla') : (\mathcal{O}_C \oplus (K(D))^\vee) \otimes E \rightarrow E$. Let us consider the bimodule structure on $\mathcal{O}_C \oplus (K(D))^\vee$ given by

$$\begin{aligned} (g, v) \cdot f &= (fg + \lambda df(v), fv) \\ f \cdot (g, v) &= (fg, fv) \end{aligned} \quad (3.7.3)$$

where f, g are local sections of \mathcal{O}_C and v is a local section of $(K(D))^\vee$ over the same open subset of C .

Then it becomes clear that requiring ∇' to satisfy equation (3.7.2) is equivalent to asking morphism $\nabla'' = (\text{Id}|_E \oplus \nabla') : (\mathcal{O}_C \oplus (K(D))^\vee) \otimes_{\mathcal{O}_X} E \rightarrow E$ to be a $(\mathcal{O}_X, \mathcal{O}_X)$ -module morphism for the previous bimodule structure of $\mathcal{O}_C \oplus (K(D))^\vee$, as then for every local sections $f, g \in \mathcal{O}_C(U)$, $v \in (K(D))^\vee(U)$ and $s \in E(U)$ over each an open set U .

$$\begin{aligned} \nabla''((g, v) \otimes (fs)) &= \nabla''(((g, v) \cdot f) \otimes s) = \nabla''((fg + \lambda df(v), fv) \otimes s) \\ &= fgs + \lambda df(v)s + f\nabla'(v \otimes s) \end{aligned}$$

From the previous explicit product formula, applying [Sim94, Theorem 2.11] it becomes clear that for each λ , the sheaf $\Lambda_1^{\text{DR}, \log D, \lambda} = \mathcal{O}_C \oplus (K(D))^\vee$ with the given bimodule structure extends to a (split quasi-polynomial) sheaf of rings of differential operators $\Lambda^{\text{DR}, \log D, \lambda}$ over C . Now let us consider the product $X = C \times \mathbb{A}^1$, and let

$p_1 : X \rightarrow C$ be the canonical projection. Then, $\Lambda_1^{\text{DR}, \log D, R} := \mathcal{O}_X \oplus p_1^*((K(D))^\vee)$ can be given a bimodule structure patching together the previous one for each value of $\lambda \in \mathbb{A}^1$. In particular, as $\mathcal{O}_X \cong \mathcal{O}_C \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{A}^1}$, we define the right action by \mathcal{O}_X as

$$((g, \lambda), v) \cdot (f \otimes \nu) = (\nu f g + \lambda \nu df(v), \nu f v) \quad (3.7.4)$$

Again, applying [Sim94, Theorem 2.11] we can extend the bimodule structure to a (split quasi-polynomial) sheaf of rings of differential operators over X flat over \mathbb{A}^1 . By construction, it coincides to the deformation to the graduate of $\Lambda^{\text{DR}, \log D}$. See [Sim94, Section 2, p. 41] for the general construction of the deformation to the graduate for a split quasi-polynomial sheaf of rings of differential operators.

Conditions (2), (3.a) and (3.b) of the definition imply that a parabolic λ -connection is a parabolic $\Lambda^{\text{DR}, \log D, R}$ -module over $\text{Spec}(\mathbb{C})$ with fixed determinant ξ . Let us fix once and for all an isomorphism $\mathcal{O}_{\mathbb{A}^1} \cong \mathbb{C}$. Let $\lambda_{x,i} = \lambda \cdot \alpha_{x,i} \in H^0(\mathbb{A}^1, \mathcal{O}_X|_{\{x\} \times \mathbb{A}^1}) \cong H^0(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1})$. Let us denote $\bar{\lambda} = \{\lambda_{x,i}\}$. Let z be a local coordinate around x . Then let

$$\begin{aligned} R_{x,i}^\lambda &= \left(-\lambda_{x,i}, \pi_C^* \left(z \frac{\partial}{\partial z} \right) \right) \in H^0(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1} \oplus \pi_C^* T_C(-D)|_{\{x\} \times \mathbb{A}^1}) \\ &= H^0(\mathbb{A}^1, \Lambda_1^{\text{DR}, \log D, R}|_{\{x\} \times \mathbb{A}^1}) \subsetneq H^0(\mathbb{A}^1, \Lambda^{\text{DR}, \log D, R}|_{\{x\} \times \mathbb{A}^1}) \end{aligned}$$

Then $\bar{R}^\lambda = \{R_{x,i}^\lambda\}$ is a residual condition for $\Lambda^{\text{DR}, \log D, R}$. From the definition, it is clear that condition (3.c) is equivalent to requiring the parabolic $\Lambda^{\text{DR}, \log D, R}$ -module to be \bar{R} -residual.

If $\nabla'' : \Lambda^{\text{DR}, \log D, R} \otimes E \rightarrow E$ is a parabolic $\Lambda^{\text{DR}, \log D, R}$ -module, it induces a parabolic $\Lambda^{\text{DR}, \log D, R}$ -module

$$\tilde{\nabla}'' : \Lambda^{\text{DR}, \log D, R} \otimes \bigwedge^r E \rightarrow \bigwedge^r E$$

Let us consider the rank one λ -connection $\lambda \cdot \nabla_{\xi, \alpha} : \xi \rightarrow \xi \otimes K(D)$ over the fixed determinant bundle. As before, it induces a $\Lambda^{\text{DR}, \log D, R}$ -module

$$\lambda \cdot \nabla''_{\xi, \alpha} : \Lambda^{\text{DR}, \log D, R} \otimes \xi \rightarrow \xi$$

Condition (3.d) is equivalent to requiring $\tilde{\nabla}''$ to coincide with morphism $\lambda \cdot \nabla''_{\xi, \alpha}$ under the isomorphism $\bigwedge^r E \cong \xi$.

$$\begin{array}{ccc} \Lambda^{\text{DR}, \log D, R} \otimes \bigwedge^r E & \xrightarrow{\tilde{\nabla}''} & \bigwedge^r E \\ \downarrow \wr & & \downarrow \wr \\ \Lambda^{\text{DR}, \log D, R} \otimes \xi & \xrightarrow{\lambda \cdot \nabla''_{\xi, \alpha}} & \xi \end{array} \quad (3.7.5)$$

Therefore, we can give the following alternative definition of a parabolic λ -connection

Definition 3.7.3. A parabolic λ -connection is a \bar{R}^λ -residual parabolic $\Lambda^{\text{DR}, \log D, R}$ -module over $\lambda : \text{Spec}(\mathbb{C}) \rightarrow \mathbb{A}^1$, $\nabla'' : \lambda^* \Lambda^{\text{DR}, \log D, R} \otimes (E, E_\bullet) \rightarrow (E, E_\bullet)$ with $\bigwedge^r E \cong \xi$ such that the induced morphism

$$\tilde{\nabla}'' : \lambda^* \Lambda^{\text{DR}, \log D, R} \otimes \bigwedge^r E \rightarrow \bigwedge^r E$$

coincides with $\lambda \cdot \nabla''_{\xi, \alpha} : \lambda^* \Lambda^{\text{DR}, \log D, R} \otimes \xi \rightarrow \xi$.

Using this equivalent definition we can prove the following theorem.

Theorem 3.7.4. Let $\mathcal{M}_{\text{Hod}}(\xi, \alpha, \bar{r}) : (\text{Sch}_{\mathbb{A}^1}) \rightarrow (\text{Sets})$ denote the functor that associates each scheme $f : T \rightarrow \mathbb{A}^1$, the set of isomorphism classes of semistable parabolic λ -connections over $C \times T$ with determinant ξ and the given parabolic type. Let $\mathcal{M}_{\text{Hod}}^s(\xi, \alpha, \bar{r})$ be the subfunctor corresponding to classes of stable parabolic λ -connections. There exists a quasi-projective variety $M_{\text{Hod}}(\xi, \alpha, \bar{r})$ such that

1. $M_{\text{Hod}}(\xi, \alpha, \bar{r})$ corepresents the functor $\mathcal{M}_{\text{Hod}}(\xi, \alpha, \bar{r})$.
2. The geometric points of $M_{\text{Hod}}(\xi, \alpha, \bar{r})$ are in bijection with the equivalence classes of semistable parabolic λ -connections with Hilbert polynomial P and the given parabolic type on C under the relation of S -equivalence.
3. There is an open subscheme $M_{\text{Hod}}^s(\xi, \alpha, \bar{r}) \subseteq M_{\text{Hod}}(\xi, \alpha, \bar{r})$ which is a coarse moduli space for the functor $\mathcal{M}_{\text{Hod}}^s(\xi, \alpha, \bar{r})$.
4. Let $m_{x,i} = r_{x,i+1} - r_{x,i}$. If the great common divisor of $\{\deg(\xi), m_{x,i} | x \in D, 1 < i \leq l_x\}$ then there is a universal stable parabolic \mathbb{A}^1 family of λ -connections $(\mathcal{E}, \mathcal{E}_\bullet, \nabla^{\text{univ}})$ over $C \times M_{\text{Hod}}^s(\xi, \alpha, \bar{r}) \times \mathbb{A}^1$. In particular, if \bar{r} corresponds to a full flag parabolic type, $M_{\text{Hod}}^s(\xi, \alpha, \bar{r})$ is a fine moduli space.

Proof. The parabolic structure \bar{r} fixes the rank of the parabolic λ -connection and ξ fixes its degree. As C is a curve, this data uniquely determines the Hilbert polynomial P of the λ -module. Let R_λ be the scheme constructed in the proof of Theorem 3.5.3 for the given Hilbert polynomial and parabolic structure, taking $\Lambda = \Lambda^{\text{DR}, \log D, R}$ over $C \times \mathbb{A}^1$ over \mathbb{A}^1 and the residual condition \bar{R}^λ . Let us consider the determinant morphism $\det : R_\lambda \rightarrow \text{Jac}(C)$ sending each geometric point of R_λ $(E, E_\bullet, \varphi, \alpha)$ to $\det(E) = \bigwedge^r E$. We denote by R_λ^ξ the pre-image of the point ξ by this morphism. Therefore, it is a closed subscheme of R_λ parameterizing locally free elements of R_λ whose determinant is ξ .

Let $f : T \rightarrow \mathbb{A}^1$ be a scheme. We say that a T -point $(E, E_\bullet, \varphi, \alpha)$ of R_λ^ξ , satisfies condition \det if the morphism

$$\tilde{\nabla}'' : f^* \Lambda^{\text{DR}, \log D, R} \otimes \bigwedge^r E \rightarrow \bigwedge^r E$$

induced by $\nabla'' : \Lambda^{\text{DR}, \log D, R} \otimes E \rightarrow E$ coincides with the morphism

$$f \cdot f^* \nabla''_{\xi, \alpha} : \pi^* \Lambda^{\text{DR}, \log D, R} \otimes \pi_C^* \xi \rightarrow \pi_C^* \xi$$

under the isomorphism $\bigwedge^r E \cong \xi$.

Let $(\mathcal{E}, \mathcal{E}_\bullet)$ be the λ -residual parabolic $\Lambda^{\text{DR}, \log D, R}$ -module underlying the universal object in R_λ^ξ . Let $\pi : R_\lambda^\xi \rightarrow \mathbb{A}^1$. The action of $\Lambda^{\text{DR}, \log D, R}$ on the universal object induces a morphism

$$\tilde{\nabla}''^{\text{univ}} : \pi^* \Lambda^{\text{DR}, \log D, R} \otimes \bigwedge^r \mathcal{E} \rightarrow \bigwedge^r \mathcal{E}$$

On the other hand, by tanking the pullback to R_λ^ξ , we have a fixed morphism

$$\pi \cdot \pi^* \nabla''_{\xi, \alpha} : \pi^* \Lambda^{\text{DR}, \log D, R} \otimes \pi_C^* \xi \rightarrow \pi_C^* \xi$$

Under the isomorphism $\bigwedge^r \mathcal{E} \cong \pi_C^* \xi$, this induces a morphism

$$\pi \cdot \pi^* \nabla''_{\xi, \alpha} : \pi^* \Lambda^{\text{DR}, \log D, R} \otimes \bigwedge^r \mathcal{E} \rightarrow \bigwedge^r \mathcal{E}$$

Then a T -point of R_λ^ξ given by $e : T \rightarrow R_\lambda^\xi$ satisfies condition det if the pullback of

$$\tilde{\nabla}''^{\text{univ}} - \pi \cdot \pi^* \nabla''_{\xi, \alpha} : \pi^* \Lambda^{\text{DR}, \log D, R} \otimes \bigwedge^r \mathcal{E} \rightarrow \bigwedge^r \mathcal{E}$$

by e is zero. By [Yok93, Lemma 4.3], there is a closed subscheme $R_\lambda^{\xi, \det}$ of R_λ^ξ such that the pullback is zero if and only if e factors through $R_\lambda^{\xi, \det}$.

Clearly, $R_\lambda^{\xi, \det}$ is a $\text{SL}_{P(N)}(\mathbb{C})$ -invariant sub-scheme of R_λ . Let $R_\lambda^{\xi, \det, ss} = R_\lambda^{\xi, \det, ss} \cap R^{ss}$ and $R_\lambda^{\xi, \det, s} = R_\lambda^{\xi, \det, ss} \cap R^s$. Then the quotient of R_λ^{ss} by $\text{SL}_{P(N)}(\mathbb{C})$ restricts to a quotient of $R_\lambda^{\xi, \det, ss}$. By Theorem 3.5.3, $R_\lambda // \text{SL}_{P(N)}(\mathbb{C})$ corepresents $\mathcal{M}(\Lambda, R, \lambda, P, \alpha, \bar{r})$, so $M_{\text{Hod}}(\xi, \alpha, \bar{r}) = R_\lambda^{LF, \xi, \det, ss} // \text{SL}_{P(N)}(\mathbb{C})$ corepresents the subfunctor of $\mathcal{M}(\Lambda, R, \lambda, P, \alpha, \bar{r})$ corresponding to locally free families with fixed determinant ξ such that diagram (3.7.5) commutes. By the previous equivalent definition, this subfunctor coincides with $\mathcal{M}_{\text{Hod}}(\xi, \alpha, \bar{r})$.

For each parabolic λ -connection (E, E_\bullet, φ) in R_λ^{ss} , the closure of the orbit by the $\text{SL}_{P(N)}(\mathbb{C})$ action coincides with the set of S-equivalent parabolic λ -connections. The closure of the orbit always contain as a representative the graduate of its Jordan-Hölder filtration $\text{Gr}(E)$, which is locally free by construction and has the same determinant bundle. Therefore, the local closures of two $\text{SL}_{P(N)}(\mathbb{C})$ -orbits in R_λ^{ss} intersect if and only if their closures intersect, because the intersection has at least a point in $R_\lambda^{\xi, \det, ss}$.

This proves that the set of closed points in $M_{\text{Hod}}(\xi, \alpha, \bar{r})$ is in correspondence with the desired set of isomorphism classes of parabolic λ -connections modulo S-equivalence.

By construction $M_{\text{Hod}}^s(\xi, \alpha, \bar{r}) \subseteq M^s(\Lambda^{\text{DR}, \log D, R}, \bar{R}^\lambda, P, \alpha, \bar{r})$. If the coprimality condition of part (4) of the theorem holds, then by Theorem 3.6.3, there exists a universal family on $M^s(\Lambda^{\text{DR}, \log D, R}, \bar{R}^\lambda, P, \alpha, \bar{r})$. Restricting it to $\mathcal{M}_{\text{Hod}}^s(\xi, \alpha, \bar{r})$ we obtain the desired universal family. \square

Now we will focus in developing the structure of the fibers of the moduli over \mathbb{A}^1 . Let λ be a closed point in \mathbb{A}^1 . Let $f : T \rightarrow \mathbb{A}^1$ be any \mathbb{A}^1 -scheme. The stability condition for parabolic Λ -modules over S is stated point-wise on the base scheme.

Therefore, by the base change formula for Λ -modules, any family of semistable parabolic $\Lambda^{\text{DR}, \log D, R}$ -modules which lie over λ , i.e., any T -point of $M_{\text{Hod}}(P, \alpha, \bar{r})$ over \mathbb{A}^1 such that the map $T \rightarrow \mathbb{A}^1$ is constant and identical to λ , is a family of semistable parabolic $\Lambda_{\lambda}^{\text{DR}, \log D, R} \cong \Lambda^{\text{DR}, \log D, \lambda}$ -modules.

Therefore, the fiber of $M_{\text{Hod}}(P, \alpha, \bar{r})$ over λ coincides with the moduli space of $\Lambda^{\text{DR}, \log D, \lambda}$ -modules. Moreover, if we restrict the right action (3.7.4) of $\mathcal{O}_X \cong \mathcal{O}_C \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{A}^1}$ on $\Lambda^{\text{DR}, \log D, R}$ to $\mathcal{O}_{\mathbb{A}^1}$, it induces an action of \mathbb{C}^* on the moduli which, by construction, preserves the fibers over \mathbb{A}^1 . From equation (3.7.4), this action coincides with the \mathbb{C}^* action

$$(E, E_{\bullet}, \nabla, \lambda) \cdot \mu = (E, E_{\bullet}, \mu \nabla, \mu \lambda)$$

The action gives an explicit isomorphism between each fiber over $\lambda \neq 0$ and the fiber over $1 \in \mathbb{A}^1$.

Let $\lambda = 0$. Then, the \mathcal{O}_C bimodule structure (3.7.3) on $\Lambda^{\text{DR}, \log D, 0}$ reduces to

$$(g, v) \cdot f = (fg, fv) = f \cdot (g, v)$$

so the left and right \mathcal{O}_C -actions are equal. As expected, $\Lambda^{\text{DR}, \log D, 0}$ coincides with the graduate of $\Lambda^{\text{DR}, \log D}$, which is simply $\Lambda^{\text{Higgs}, \log D}$. The residue restricts to $X_0 \cong C$ as the null section, so the fiber of the moduli over $\lambda = 0$ is a family of 0-residual parabolic $\Lambda^{\text{Higgs}, \log D}$ -modules satisfying condition Therefore, the fiber over $\lambda = 0$ coincides with the moduli space of parabolic Higgs bundles over C . On the other hand, from (3.7.3) it is clear that $\Lambda^{\text{DR}, \log D, 1} \cong \Lambda^{\text{DR}, \log D}$, so the fiber over $\lambda = 1$ (and therefore, over any nonzero λ) is isomorphic to the moduli space of vector bundles with a parabolic connection.

3.8 Riemann-Hilbert correspondence for parabolic connections

In this section we will study an analogue of the Riemann-Hilbert correspondence for parabolic connections. In order to set up the basis, let us first review the construction of the classical Riemann-Hilbert map. Let X be any variety and let us fix a base point $x \in X$. Given a connection (E, ∇) over X , for each loop $\gamma : [0, 1] \rightarrow X$ starting and ending in x , the parallel transport along γ for the connection ∇ induces a linear map $\rho_{(E, \nabla)}(\gamma) : E|_x \rightarrow E|_x$. If ∇ is a flat connection, then the map $\rho_{(E, \nabla)}(\gamma)$ does only depend on the homotopy class of the loop γ . Therefore, for each integrable connection (E, ∇) over X we obtain a map

$$\rho_{(E, \nabla)} : \pi_1(X, x) \longrightarrow \text{GL}(E|_x)$$

By construction, it is a representation of the fundamental group. Moreover, if we fix the rank of E , then $E|_x \cong \mathbb{C}^r$, so we obtain a representation $\rho_{(E, \nabla)} : \pi_1(X, x) \longrightarrow \text{GL}_r(\mathbb{C})$. The isomorphism $E|_x \cong \mathbb{C}^r$ is not canonical, so $\rho_{(E, \nabla)}$ is only well defined up to conjugation by an automorphism of \mathbb{C}^r . Notice that we change the base point $x \in X$ to another $y \in X$, parallel transport gives us a way to identify the fibers $E|_x$ and $E|_y$, so the new map $\pi_1(X, y) \longrightarrow \text{GL}_r(\mathbb{C})$ can be obtained from the other

one by conjugating through the isomorphism $E|_x \cong E|_y$. Therefore, for each flat connection (E, ∇) we obtain an element of $\text{Hom}(\pi_1(X), \text{GL}_r(\mathbb{C})) / \text{GL}_r(\mathbb{C})$.

This process can be reversed. Take a representation $\rho : \pi_1(X) \rightarrow \text{GL}_r(\mathbb{C})$. Let \tilde{X} be the universal cover of X . Then $\pi_1(X)$ acts on $\tilde{X} \times \mathbb{C}^r$ through the representation ρ . The quotient $E = (\tilde{X} \times \mathbb{C}^r) / \sim_\rho$ is a vector bundle of rank r over E . Giving $\tilde{X} \times \mathbb{C}^r$ the canonical differential $d : \mathbb{C}^r \rightarrow \mathbb{C}^r \otimes \Omega^1(\tilde{X})$, the quotient E inherits a flat connection $\nabla : E \rightarrow E \otimes \Omega^1(X)$. Moreover it is clear that this correspondence maps irreducible connections to irreducible representations. In the case where X is a compact connected Riemann surfaces, a connection is irreducible if and only if it is stable. Moreover, irreducible representations in $\text{Hom}(\pi_1(X), \text{GL}_r(\mathbb{C}))$ are also GIT-stable for the action of $\text{GL}_r(\mathbb{C})$ under conjugation. Therefore, if $\mathcal{M}_{\text{DR}}(X, r)$ denotes the moduli space of stable flat connections on X of rank r and $\mathcal{M}_B(X, r) = \text{Hom}(\pi_1(X), \text{GL}_r(\mathbb{C})) // \text{GL}_r(\mathbb{C})$ denotes the moduli space of representations of the fundamental group on $\text{GL}_r(\mathbb{C})$, then there is a bijective map

$$\text{RH}_X : \mathcal{M}_{\text{DR}}(X, r) \rightarrow \mathcal{M}_B(X, r)$$

known as Riemann-Hilbert correspondence. Moreover, it is a well known result that this map is actually a biholomorphism. Observe that the construction of the moduli space $\mathcal{M}_{\text{DR}}(X, r)$ depends on a choice of an algebraic structure on X , while space of representations $\mathcal{M}_B(X, r)$ does only depend on the topological type of X , i.e., its genus. In particular, we know that the biholomorphic class of the moduli space of flat connections does not depend on the isomorphism class of the curve X (only on its topological type). Nevertheless, it is interesting to notice that the isomorphism class of $\mathcal{M}_{\text{DR}}(X, r)$ as an algebraic variety does depend on the isomorphism class of the curve and, in fact, as we saw in Chapter 2, we have a Torelli type theorem for this moduli space due to Biswas and Muñoz [BM07]. In particular, this implies that the map RH_X is a biholomorphism, but not an algebraic isomorphism.

The parabolic analogue of the previous correspondence was described by Simpson [Sim90]. Let X be a compact connected Riemann surface and let D be finite set of points in X . Simpson proved that there exists a categorical correspondence between the following three categories

- Stable parabolic Higgs bundles (E, E_\bullet, φ) , where $\varphi \in H^0(\text{PEnd}(E, E_\bullet) \otimes K(D))$
- Stable parabolic connections (V, V_\bullet, ∇) , where $\nabla : V \rightarrow V \otimes K(D)$ is a logarithmic connection whose residue at each parabolic point preserves the filtration
- Stable filtered local systems (L, L_\bullet) , where L is a local system over $X \setminus D$ and L_\bullet is a filtration by subsheaves over each stalk L_{ρ_x} of L at a some ray ρ_x emanating from each parabolic point $x \in D$

Moreover, there exists an explicit relation between the parabolic weights and eigenvalues of the residue at each parabolic point between the corresponding objects in each category, in the sense that if (E, E_\bullet, φ) is a stable parabolic Higgs bundle with parabolic weights $\alpha = \{\alpha_j(x)\}$ and whose residue at the parabolic points $x \in D$ has eigenvalues $\{b_j(x) + c_j(x)i\}$ then the associated stable parabolic connection

and stable filtered local system have the following prescribed parabolic weights and eigenvalues of their corresponding residues [Sim90, page 720]

	(E, E_\bullet, φ)	(V, V_\bullet, ∇)	(L, L_\bullet)
weight	$\alpha_j(x)$	$\alpha_j(x) - 2b_j(x)$	$-2b_j(x)$
eigenvalue	$b_j(x) + c_j(x)i$	$\alpha_j(x) + 2c_j(x)i$	$\exp(-2\pi i\alpha_j(x) + 4\pi c_j(x))$

(3.8.1)

Observe that if $b_j(x) = 0$ for all j , then the parabolic weights for (L, L_\bullet) at x are all zero. Therefore, the parabolic structure of (L, L_\bullet) is trivial (it consists of the only possible dimension r jump at $\beta_1(x) = 0$). If this happens for every $x \in D$, then (L, L_\bullet) is uniquely determined by the local system L over $X \setminus D$ and, therefore, it is fully determined by the associated representation of the fundamental group of $X \setminus D$. In particular, if (E, E_\bullet, φ) is a strongly parabolic Higgs bundle then we obtain then the residue of φ is nilpotent and, thus, we obtain the following restricted correspondence

	(E, E_\bullet, φ)	(V, V_\bullet, ∇)	(L, L_\bullet)
weight	$\alpha_j(x)$	$\alpha_j(x)$	0
eigenvalue	0	$\alpha_j(x)$	$\exp(-2\pi i\alpha_j(x))$

(3.8.2)

Thus, we obtain a correspondence between parabolic connections whose eigenvalues of the monodromy coincide with the parabolic weights $\alpha_j(x)$ and representations of $\pi_1(X)$ such that the image of a loop around each $x \in D$ has eigenvalues $e^{-2\pi i\alpha_j(x)}$. Observe that, in general, the resulting representation does not live in $\mathrm{SL}_r(\mathbb{C})$, even if we fixed the determinant of the connection (E, ∇) . The Simpson correspondence commutes with determinants, so in order to obtain a proper $\mathrm{SL}_r(\mathbb{C})$ -representation it will be enough to adjust the determinant of (E, ∇) and impose a restriction on the weights α forcing the resulting eigenvalues of the representation $e^{-2\pi i\alpha_j(x)}$ to have product 1. As we will see later on, the motivation behind trying to set a correspondence explicitly with moduli spaces of $\mathrm{SL}_r(\mathbb{C})$ -representations is that we are interested in building an analogue of the Deligne–Hitchin moduli space for parabolic connections with fixed determinant. In order to do so, we want to prove that the restriction of the Simpson correspondence to these choices of parameters actually give a biholomorphism between the corresponding moduli space, thus allowing the gluing in the Deligne–Hitchin construction.

I must mention that the regularity result presented here has already been proved (in fact, in a more general context) by Inaba in [Ina13]. Nevertheless, the proof given here was developed independently and it is, in a sense, more explicit.

Before stating the actual conditions on the determinant and parabolic weights and engaging in the main regularity theorem, we shall introduce some notation and lemmata about the spectrum of holomorphic families of matrices and holomorphic matrix-valued maps. Given a matrix A , let $\lambda(A)$ denote the spectrum of A .

Lemma 3.8.1. *Let $\Psi : \Omega \rightarrow \mathbb{C}$ be a holomorphic function defined on a complex domain $\Omega \subseteq \mathbb{C}$ and let $\mathrm{GL}(r, \mathbb{C})^\Omega$ be the open subvariety of $\mathrm{GL}(r, \mathbb{C})$ parameterizing matrices $A \in \mathrm{GL}(r, \mathbb{C})$ such that $\lambda(A) \subseteq \Omega$. Then Ψ induces an holomorphic morphism $\bar{\Psi} : \mathrm{GL}(r, \mathbb{C})^\Omega \rightarrow \mathrm{GL}(r, \mathbb{C})$ that commutes with the action of $\mathrm{GL}(r, \mathbb{C})$ by conjugation.*

Corollary 3.8.2. *Let L be a holomorphic rank r vector bundle over a complex manifold T and let $\alpha_1, \dots, \alpha_r$ be distinct complex numbers. Let A be a section of $\mathrm{GL}(L)$ such that for each $t \in T$, $\lambda(A(t)) = \{\alpha_1, \dots, \alpha_r\}$. Let $\Psi : \Omega \rightarrow \mathbb{C}$ be a holomorphic function defined on a complex domain $\Omega \subseteq \mathbb{C}$ such that $\alpha_i \in \Omega$ for $i = 1, \dots, r$. Then $\bar{\Psi}(A)$ is a holomorphic section of $\mathrm{GL}(L)$.*

Proof. Let us fix a trivialization of L over a cover $\{T_i\}_{i \in I}$ of T . Then the section A can be locally expressed over T_i as a matrix-valued holomorphic function $A_i : T_i \rightarrow \mathrm{GL}(r, \mathbb{C})$. As the spectrum is preserved by conjugation, by hypothesis, $A_i(t)$ belongs to $\mathrm{GL}(r, \mathbb{C})^\Omega$ for every $t \in T_i$, so $A_i(t)$ factors through $A_i : T_i \rightarrow \mathrm{GL}(r, \mathbb{C})^\Omega$. Now, $\bar{\Psi}$ defines a holomorphic morphism $\mathrm{GL}(r, \mathbb{C})^\Omega \rightarrow \mathrm{GL}(r, \mathbb{C})$. Composing yields a holomorphic morphism $\bar{\Psi}(A_i) : T_i \rightarrow \mathrm{GL}(r, \mathbb{C})$ for every $i \in I$. As conjugation of A_i commutes with $\bar{\Psi}$, if we consider another open subset of the cover T_j and we consider the conjugation by transition functions $\varphi_{ij} : T_i \cap T_j \rightarrow \mathrm{GL}(r, \mathbb{C})$, then

$$A_j = \varphi_{ij} A_i \varphi_{ij}^{-1}$$

so

$$\bar{\Phi}(A_j) = \bar{\Phi}(\varphi_{ij} A_i \varphi_{ij}^{-1}) = \varphi_{ij} \bar{\Phi}(A_i) \varphi_{ij}^{-1}$$

Therefore, we can define a global holomorphic section $\bar{\Phi}(A) : T \rightarrow \mathrm{GL}(L)$. \square

Lemma 3.8.3. *Let M and N be smooth complex analytic varieties and let $f : M \rightarrow N$ be a holomorphic one to one map. Then f is a biholomorphism.*

Proof. As f is one to one, there exists a pointwise inverse function $f^{-1} : N \rightarrow M$. In order to prove that it is holomorphic, it is only necessary to prove that it is locally holomorphic. Let $p \in M$ and $q = f(p) \in N$. Let (V_q, ψ_q) be a holomorphic chart for N around q . Let $U = f^{-1}(V_q)$. Let (U_p, φ_p) be a holomorphic chart for M around p and let us consider $W = U \cap U_p$. Then f defines a holomorphic function

$$\psi_q \circ f \circ \varphi_p^{-1} : \varphi_p(W) \longrightarrow \psi_q(V_q) \subset \mathbb{C}^n$$

By the open mapping theorem, $\psi_q \circ f \circ \varphi_p^{-1}$ is open, so $f(W) \subset V_q$ is an open neighborhood of q and $f|_W : W \rightarrow f(W)$ is a holomorphic homomorphism. In particular, this implies that the charts must have the same dimension. Then

$$\psi_q|_{f(W)} \circ f \circ \varphi_p^{-1}|_{\varphi_p(W)} : \varphi_p(W) \longrightarrow \varphi_q(f(W))$$

is a one to one holomorphic map between open subsets of \mathbb{C}^n . By [GH11, p.19] its inverse map $\varphi_p|_{\varphi_p(W)} \circ f^{-1} \circ \psi_q^{-1}|_{f(W)}$ is holomorphic. Therefore, $f^{-1}|_{f(W)} : f(W) \rightarrow W$ is a holomorphic map.

As f is bijective, we have found an open neighborhood $f(W)$ for each $q \in N$, such that $f^{-1}|_{f(W)}$ is holomorphic. Therefore, $f^{-1} : M \rightarrow N$ is a holomorphic map. \square

Lemma 3.8.4. *Let \mathcal{M} and \mathcal{M}' be coarse moduli spaces in the category of smooth holomorphic manifolds for the contravariant functors $\mathcal{F}, \mathcal{F}' : (\mathrm{Hol}) \rightarrow (\mathrm{Sets})$ respectively. If there is a natural transformation $\alpha : \mathcal{F} \rightarrow \mathcal{F}'$ such that $\alpha(\{\mathrm{pt}\}) : \mathcal{F}(\{\mathrm{pt}\}) \rightarrow \mathcal{F}'(\{\mathrm{pt}\})$ is bijective, then there exists a biholomorphism between \mathcal{M} and \mathcal{M}' .*

Proof. Let $\psi : \mathcal{F} \rightarrow \text{Hom}(-, \mathcal{M})$ and $\psi' : \mathcal{F}' \rightarrow \text{Hom}(-, \mathcal{M}')$ be natural transformations such that (\mathcal{M}, ψ) and (\mathcal{M}', ψ') co-represent \mathcal{F} and \mathcal{F}' respectively. Then $\psi' \circ \alpha : \mathcal{F} \rightarrow \text{Hom}(-, \mathcal{M}')$ is a natural transformation. By the universal property of coarse moduli spaces, there exists a unique holomorphic morphism $f : \mathcal{M} \rightarrow \mathcal{M}'$ such that $\psi' \circ \alpha = f^\# \circ \psi$, where $f^\# : \text{Hom}(-, \mathcal{M}) \rightarrow \text{Hom}(-, \mathcal{M}')$ is the natural transformation given by

$$\begin{aligned} f^\# : \text{Hom}(T, \mathcal{M}) &\longrightarrow \text{Hom}(T, \mathcal{M}') \\ g &\longmapsto f \circ g \end{aligned}$$

Evaluating all natural transformations at a point yields

$$\psi'(\{\text{pt}\}) \circ \alpha(\{\text{pt}\}) = f^\#(\{\text{pt}\}) \circ \psi(\{\text{pt}\})$$

By coarse moduli axioms, both $\psi(\{\text{pt}\})$ and $\psi'(\{\text{pt}\})$ are bijective, so

$$f^\#(\{\text{pt}\}) = \psi'(\{\text{pt}\}) \circ \alpha(\{\text{pt}\}) \circ \psi(\{\text{pt}\})^{-1}$$

By hypothesis, $\alpha(\{\text{pt}\})$ is bijective too, so $f^\#(\{\text{pt}\})$ is bijective. Identifying \mathcal{M} and \mathcal{M}' with $\text{Hom}(\{\text{pt}\}, \mathcal{M})$ and $\text{Hom}(\{\text{pt}\}, \mathcal{M}')$ respectively, this implies that f is bijective.

Then, $f : \mathcal{M} \rightarrow \mathcal{M}'$ is a holomorphic and bijective map between complex analytic varieties. By Lemma 3.8.3, f is a biholomorphism. \square

Before proving the main theorem we will need the following lemma in order to prove that the parabolic structure of a family of full flag parabolic connections is uniquely determined by the residue of the connection along the parameter space.

Lemma 3.8.5. *Let E be a vector bundle of rank r over a scheme or a complex manifold T . Let $f : E \rightarrow E$ be a vector bundle morphism such that for every t , $f_t : E_t \rightarrow E_t$ has r different fixed eigenvalues $\alpha_1, \dots, \alpha_r$. Then there exists a unique filtration of E by subbundles*

$$E = E_0 \supsetneq E_1 \supsetneq \dots \supsetneq E_r = 0$$

such that f preserves the filtration and f acts on E_{i-1}/E_i as multiplication by α_i for all $i = 1, \dots, r$.

Proof. As f_t has r fixed eigenvalues $\alpha_1, \dots, \alpha_r$, for each $i = 1, \dots, r$ for each $t \in T$, the map $f - \alpha_i \text{Id} : E \rightarrow E$ has constant rank $r - 1$, so it is a morphism of vector bundles and $\text{Im}(f - \alpha_i \text{Id})$ is a subbundle of E of rank $r - 1$ for all i . Similarly, as the eigenspaces of f_t for different eigenvalues have pairwise trivial intersection, the composition map $F_k := \prod_{i=1}^k (f - \alpha_i \text{Id}) : E \rightarrow E$ has constant rank $r - k$, so it is a morphism of vector bundles and $E_k := \text{Im}(F_k)$ is a subbundle of E of rank $r - k$ for all k . Moreover, as $F_{k+1} = (f - \alpha_{k+1} \text{Id}) \circ F_k$, yields

$$\begin{aligned} E_{k+1} &= \text{Im}(F_{k+1}) = \text{Im}((f - \alpha_{k+1} \text{Id}) \circ F_k) = \text{Im}((f - \alpha_{k+1} \text{Id})|_{\text{Im}(F_k)}) \\ &= \text{Im}((f - \alpha_{k+1} \text{Id})|_{E_k}) \subseteq E_k \end{aligned}$$

The inclusion is strict because $\text{rk}(E_{k+1}) = r - k - 1 < r - k = \text{rk}(E_k)$. Moreover, as $E_{k+1} = \text{Im}((f - \alpha_{k+1} \text{Id})|_{E_k})$, this implies that $f - \alpha_{k+1} \text{Id}$ acts as the zero morphism in E_k/E_{k+1} for each k , so f acts as multiplication by α_{k+1} . Therefore, $\{E_k\}$ gives the desired filtration.

Now, let $E = E'_0 \supseteq E'_1 \supseteq \dots \supseteq E'_r = 0$ be any filtration of E satisfying the conditions of the lemma. Let us prove that $\{E'_k\}$ must coincide with the previous filtration.

As f preserves the filtration and acts on the quotient by multiplication by α_k , the image of the restricted morphism $f|_{E'_k} - \alpha_k \text{Id} : E'_{k-1} \rightarrow E'_{k-1}$ lies in E'_k for every $k = 1, \dots, r$. Therefore, for every $k = 1, \dots, r$

$$\text{Im}((f - \alpha_k \text{Id})|_{E'_{k-1}}) \subseteq E'_k$$

Let us prove by induction on k that $E'_k = E_k$. For $k = 0$, $E'_k = E = E_k$. Suppose that the equality holds for $k' < k$. Then

$$E'_k \supseteq \text{Im}((f - \alpha_k \text{Id})|_{E'_{k-1}}) = \text{Im}((f - \alpha_k \text{Id})|_{E_{k-1}}) = E_k$$

As both filtrations are full flag, $\text{rk}(E_k) = \text{rk}(E'_k) = r - k$, so $E_k = E'_k$. \square

Lemma 3.8.6. *Let \mathcal{C} be any category. Let $F : \mathcal{C} \rightarrow (\text{Sets})$ and $G : \mathcal{C} \rightarrow (\text{Sets})$ be two contravariant functors from \mathcal{C} to the category of sets. Let $\Phi : F \rightarrow G$ be a natural transformation such that $\Phi(T)$ is bijective for every $T \in \mathcal{C}$. Then Φ is an isomorphism of functors.*

Proof. If $\Phi(T) : F(T) \rightarrow G(T)$ is bijective then for every $T \in \mathcal{C}$ there exists an inverse $\Phi^{-1}(T) : G(T) \rightarrow F(T)$. It is only necessary to prove that Φ^{-1} is functorial. Let $S \in \mathcal{C}$ and let $f \in \text{Hom}(S, T)$. We have to prove that the following diagram commutes

$$\begin{array}{ccc} F(T) & \xleftarrow{\Phi^{-1}(T)} & G(T) \\ \downarrow F(f) & & \downarrow G(f) \\ F(S) & \xleftarrow{\Phi^{-1}(S)} & G(S) \end{array}$$

As $\Phi(S) : F(S) \rightarrow G(S)$ is a bijection, for each $t' \in G(T)$, $(F(f) \circ \Phi^{-1}(T))(t') = (\Phi^{-1}(S) \circ G(f))(t')$ if and only if their images under $\Phi(S)$ coincide, i.e., if

$$(\Phi(S) \circ F(f) \circ \Phi^{-1}(T))(t') = (\Phi(S) \circ \Phi^{-1}(S) \circ G(f))(t') = G(f)(t')$$

The diagram is commutative if and only if the previous equation holds for every $t' \in G(T)$. As $\Phi(T) : F(T) \rightarrow G(T)$ is a bijection, for every $t' \in G(T)$ there exists an object $t \in F(t)$ such that $t' = \Phi(T)(t)$, so testing commutativity against all object $t' \in G(T)$ is equivalent to testing it against the images under $\Phi(T)$ of every object $t \in F(T)$. Therefore, we have to prove that for every $t \in F(T)$,

$$\begin{aligned} (\Phi(S) \circ F(f))(t) &= (\Phi(S) \circ F(f) \circ \Phi^{-1}(T))(\Phi(T)(t)) \\ &= (\Phi(S) \circ F(f) \circ \Phi^{-1}(T))(t') = G(f)(t') = (G(f) \circ \Phi(T))(t) \end{aligned}$$

The previous equality holds because Φ is a natural transformation and, therefore, it's functorial. \square

Let X be a Riemann surface. Let $D = \{x_1, \dots, x_n\}$ be a set of $n \geq 1$ distinct points over X and let $\alpha = \{\alpha_i(x)\}$ be a generic full flag system of parabolic weights of rank r over D such that for every $x \in D$,

$$\beta(x) := \sum_{i=1}^r \alpha_i(x) \in \mathbb{Z}$$

Let us consider the following holomorphic line bundle over X

$$\xi := \mathcal{O}_X \left(\sum_{x \in D} \beta(x)x \right)$$

Considering ξ as a parabolic vector bundle with the trivial filtration on the parabolic points and parabolic weights $\beta = \{\beta(x)\}$, the Simpson correspondence [Sim90] between filtered Higgs bundles, filtered \mathcal{D}_X -modules and filtered local systems of X implies the existence of a parabolic connection $\nabla_{\xi, \beta} : \xi \rightarrow \xi \otimes \Omega^1(\log(D))$ on ξ corresponding both to the zero Higgs field on ξ and the constant filtered local system on X with trivial filtrations and parabolic system of weights β .

Let $\mathcal{M}_{DR}(X, \xi, \alpha)$ denote the moduli space of semistable triples (E, E_\bullet, ∇) consisting on a holomorphic parabolic vector bundle (E, E_\bullet) of rank r , and determinant ξ , together with an irreducible flat parabolic connection $\nabla : E \rightarrow E \otimes \Omega^1(\log(D))$ over X satisfying

1. If f is a locally defined holomorphic function on X and s is a locally defined holomorphic section of E then

$$\nabla(fs) = f \cdot \nabla(s) + \lambda \cdot s \otimes df$$

2. For each $x \in D$ the homomorphism induced in the filtration over the fiber E_x satisfies

$$\nabla(E_{x,i}) \subseteq E_{x,i} \otimes K(D)|_x$$

3. For every $x \in D$ and every $i = 1, \dots, r$ the action of $\text{Res}(\nabla, x)$ on $E_{x,i}/E_{x,i-1}$ is the multiplication by $\alpha_i(x)$. Since $\text{Res}(\nabla, x)$ preserves the filtration, it acts on each quotient.
4. The operator $\text{tr}(\nabla) : \bigwedge^r E \rightarrow (\bigwedge^r E) \otimes K(D)$ induced by ∇ coincides with $\nabla_{\xi, \beta}$.

On the other hand, let $U = X \setminus D$. Fix a point $x_0 \in U$. For every $x \in D$, let $\gamma_x \in \pi_1(U, x_0)$ be the class of a positively oriented simple loop around x . Let $\mathcal{M}_B(X, \alpha)$ be the subvariety of $\text{Hom}(\pi_1(U, x_0), \text{SL}(r, \mathbb{C})) // \text{SL}(r, \mathbb{C})$ corresponding to classes of irreducible representations $\rho : \pi_1(U, x_0) \rightarrow \text{SL}(r, \mathbb{C})$ such that for each $x \in D$, $\rho(\gamma_x)$ has eigenvalues $\{e^{-2\pi i \alpha_i(x)}\}$. The group $\text{SL}(r, \mathbb{C})$ acts on $\text{Hom}(\pi_1(U, x_0), \text{SL}(r, \mathbb{C}))$ through the adjoint action of $\text{SL}(r, \mathbb{C})$ on itself. Since the eigenvalues of $\rho(\gamma_x)$ are preserved by conjugation, the quotient is well defined. On the other hand, the determinant of $\rho(\gamma_x)$ is the product of its eigenvalues, so

$$\det(\rho(\gamma_x)) = \prod_{i=1}^r e^{-2\pi i \alpha_i(x)} = e^{-2\pi i \sum_{i=1}^r \alpha_i(x)} = 1$$

The fundamental groups for different base points are identified up to an inner automorphism and the different choices of the loops γ_x are identified through an outer isomorphism. Thus, the isomorphism class of the space $\mathcal{M}_B(X, \alpha)$ is independent of the choice of x_0 and the loops γ_x , so we can omit any reference to both of them.

The moduli space $\mathcal{M}_{DR}(X, \xi, \alpha)$ is a coarse moduli space for the functor of families of parabolic semistable flat connections on X in the category of holomorphic varieties, i.e., the functor $\text{Hom}(\bullet, \mathcal{M}_{DR}(X))$ corepresents the contravariant functor $\mathcal{F}_{DR} : (\text{Hol}) \rightarrow (\text{Sets})$ that sends each holomorphic variety T to the set of isomorphism classes of triples consisting on a holomorphic vector bundle E over $X \times T$, a decreasing full flag filtration by subbundles $E_{x,i}$ over $\{x\} \times T$ for each $x \in D$ and an irreducible relative logarithmic connection $\nabla : E \rightarrow E \otimes \Omega_{X/T}^1(\log(D \times T))$, such that for every $t \in T$, $(E_t, (E_\bullet)_t, \nabla_t)$ is semistable, flat over X and satisfies properties (1) to (4) in the definition of parabolic connection,

$$\mathcal{F}_{DR}(T) = \left\{ \begin{array}{ll} (E, E_\bullet, \nabla) & \nabla : E \rightarrow E \otimes \Omega_{X/T}(\log(D \times T)) \text{ is a relative} \\ \downarrow & \text{logarithmic connection such that } (E_t, (E_\bullet)_t, \nabla|_{E_t}) \\ X \times T & \text{is s.s. flat over } X \text{ and satisfies (1)-(4) for all } t \in T \end{array} \right\} / \sim$$

Similarly, given a fixed point $x_0 \in U$ and classes of positively oriented loops γ_x around x for each $x \in D$, the moduli space $\mathcal{M}_B(X, \alpha)$ is a coarse moduli space in the category of holomorphic varieties for the functor of families of representations of the group $\pi_1(U)$ in $\text{SL}(r, \mathbb{C})$ such that the image of γ_x has prescribed eigenvalues $\{e^{-2\pi i \alpha_i(x)}\}$ modulo conjugation, i.e., it corepresents the contravariant functor $\mathcal{F}_B : (\text{Hol}) \rightarrow (\text{Sets})$ that sends each complex variety T to the set of isomorphism classes of holomorphic vector bundles L over T together with a morphism $\rho : \pi_1(X) \times T \rightarrow \text{SL}(L)$ such that for each $t \in T$, $\rho_t : \pi_1(U) \rightarrow \text{SL}(L)$ lies in $\text{SL}(L_t)$, it is a group homomorphism and $\rho_t(\gamma_x)$ has eigenvalues $\{e^{-2\pi i \alpha_i(x)}\}$, modulo the action of $\text{SL}(L)$ by conjugation, i.e., if $\psi : T \rightarrow \text{SL}(L)$ is a section of $\text{SL}(L)$ and we denote by $\psi^{(-1)} : T \rightarrow \text{SL}(L)$ the section $\psi^{(-1)}(t) = \psi(t)^{-1}$, then $\rho \sim \psi \rho \psi^{(-1)}$.

$$\mathcal{F}_B(T) = \left\{ \begin{array}{ll} L & \\ \downarrow & \text{is a holomorphic vector bundle and} \\ (L, \rho): & T \\ \rho : \pi_1(U) \times T \rightarrow \text{SL}(L) & \text{is irreducible holomorphic on } T \end{array} \right\} /_{\text{SL}(L)}$$

Theorem 3.8.7. *Let X be a Riemann surface. Let $D = \{x_1, \dots, x_n\}$ be a set of $n \geq 1$ distinct points over X and let $\alpha = \{\alpha_i(x)\}$ be a full flag generic system of parabolic weights of rank r over D such that for every $x \in D$,*

$$\beta(x) := \sum_{i=1}^r \alpha_i(x) \in \mathbb{Z}$$

Let us consider the following holomorphic line bundle over X

$$\xi := \mathcal{O}_X \left(\sum_{x \in D} \beta(x) x \right)$$

Then the monodromy map (Riemann-Hilbert map) defines a biholomorphic correspondence between $\mathcal{M}_{DR}(X, \xi, \alpha)$ and $\mathcal{M}_B(X, \alpha)$.

Proof. As α is full flag and generic, both $\mathcal{M}_{DR}(X, \xi, \alpha)$ and $\mathcal{M}_B(X, \alpha)$ are smooth, and therefore, they are complex analytic varieties. We will prove that the monodromy map defines a natural transformation between \mathcal{F}_{DR} and \mathcal{F}_B which is bijective at the level of points. As $\mathcal{M}_{DR}(X, \xi, \alpha)$ and $\mathcal{M}_B(X, \alpha)$ co-represent the functors \mathcal{F}_{DR} and \mathcal{F}_B respectively, by Lemma 3.8.4, this natural transformation induces a biholomorphism between $\mathcal{M}_{DR}(X, \xi, \alpha)$ and $\mathcal{M}_B(X, \alpha)$.

Let $x_0 \in U$ be a fixed point. Moreover, fix once and for all a finite set of generators of $\pi_1(U) \cong \pi_1(U, x_0) = \langle \gamma_i \rangle_{i=1}^m$ including the class γ_x of a small loop around x for each $x \in D$.

For each holomorphic variety T , let us consider the map $\Phi(T) : (\text{Sets}) \rightarrow (\text{Sets})$ that sends each family (E, E_\bullet, ∇) over T , to the pair consisting on the vector bundle $E|_{\{x_0\} \times T}$ and the conjugacy class of the representation

$$\begin{aligned} \rho_{(E, E_\bullet, \nabla)} & : \quad \pi_1(U, x_0) \times T \longrightarrow \text{GL}(E|_{\{(x_0, t)\}}) \\ (\gamma, t) & \longmapsto e^{-\int_\gamma \nabla_t} \end{aligned}$$

For each $t \in T$ the morphism corresponds to the monodromy of the flat parabolic connection $\nabla_t := \nabla|_{E_t} : E_t \rightarrow E_t \otimes K_X(D)$ around the loop γ , so it is well defined and it is a group homomorphism. Let us prove that the obtained representation belongs to $\mathcal{F}_B(T)$.

Let $\tilde{\gamma}_x$ denote a small positively oriented loop around $x \in D$ contained within an open set of X over which E is trivial. Choosing a local coordinate centered in x and a basis for E over this open set, ∇_t can be locally expressed as $\nabla_t = A_\nabla dz/z + d$. As $\tilde{\gamma}_x$ is closed, integrating along it yields

$$e^{-\int_{\tilde{\gamma}_x} \nabla_t} = e^{-\int_{\tilde{\gamma}_x} A_\nabla dz/z} = e^{-2\pi i A_\nabla} = e^{-2\pi i \text{Res}(\nabla_t, x)}$$

By hypothesis, $\lambda(\text{Res}(\nabla_t, x)) = \{\alpha_i(x)\}_{i=1}^r$. Therefore,

$$\lambda\left(e^{-\int_{\tilde{\gamma}_x} \nabla_t}\right) = \lambda\left(e^{-2\pi i \text{Res}(\nabla_t, x)}\right) = e^{-2\pi i \lambda(\text{Res}(\nabla_t, x))} = \{e^{-2\pi i \alpha_i(x)}\}$$

In particular, as $\sum_{i=1}^r \alpha_i(x) \in \mathbb{Z}$ and all the $\alpha_i \in [0, 1)$ are different, yields

$$\det\left(e^{-\int_{\tilde{\gamma}_x} \nabla_t}\right) = \prod_{i=1}^r e^{-2\pi i \alpha_i(x)} = e^{-2\pi i \sum_{i=1}^r \alpha_i(x)} = 1$$

Now, if we obtain a representative of γ_x by deforming the loop $\tilde{\gamma}_x$ so that it starts in x_0 , then $e^{-\int_{\gamma_x} \nabla_t} = \rho_{(E, E_\bullet, \nabla)}(\gamma_x, t)$ belongs to the conjugacy class of $e^{-\int_{\tilde{\gamma}_x} \nabla_t}$, so its spectrum is the same. In particular, it belongs to $\text{SL}(r, \mathbb{C})$.

On the other hand, Simpson proved that the map that sends a filtered \mathcal{D}_X -modules on X to the filtered local system of its solutions on U define an equivalence of categories between the category of stable filtered \mathcal{D}_X -modules and the category of stable filtered local systems [Sim90]. Moreover, this category equivalence preserves determinants and establishes the fixed correspondence between the parabolic weights

and eigenvalues of the residue of the filtered \mathcal{D}_X -module and the parabolic weights and eigenvalues of the residue of the filtered local system shown in table (3.8.1).

A parabolic connection in our definition corresponds to a class of filtered \mathcal{D}_X -modules in which we have fixed the system of weights and taken the spectrum of the residue of the connection to be real and equal to the parabolic weights. As we showed in table (3.8.2) the correspondent filtered local system must have all parabolic weights equal to zero, therefore having a trivial filtration. Moreover, in this case stability condition for filtered local systems is equivalent to the irreducibility of the local system.

The category of filtered local systems with a trivial filtration is clearly equivalent to the category of local systems on U . Fixing a point $x_0 \in U$ allows us to canonically identify the latter with the category of the conjugacy classes of representations of the fundamental group of U in $\mathrm{GL}(r, \mathbb{C})$. Under this correspondence, which is also compatible with determinants, irreducible local systems correspond to irreducible representations.

Finally, by property (4) in the definition, every element in $\mathcal{M}_{DR}(X, r, \alpha)$ has the same determinant, namely $(\xi, \xi_\bullet, \nabla_{\xi, \beta})$. This element corresponds under the Simpson correspondence to the trivial rank one local system and, therefore, to the constant representation $\rho(\gamma) = I$ for all $\gamma \in \pi_1(U, x_0)$. The representations whose determinant is the constant representation are precisely those that factor through $\mathrm{SL}(r, \mathbb{C})$.

By construction, the composition of Simpson correspondence and the correspondence between local systems and representations of the fundamental group coincide with the previously defined monodromy map. Therefore, as the Simpson correspondence is compatible with determinants, the image of the elements in $\mathcal{M}_{DR}(X, \xi, \alpha)$ under the monodromy map is always an irreducible representation of $\pi_1(U, r)$ in $\mathrm{SL}(r, \mathbb{C})$.

This fact, together with the previous consideration on the spectrum of the images proves that the monodromy map is an equivalence of categories between the category of parabolic flat connections in $\mathcal{M}_{DR}(X, \xi, \alpha)$ and the category of representations of the fundamental group in $\mathcal{M}_B(X, \alpha)$. Therefore, $\rho_{(E, E_\bullet, \nabla)}(\cdot, t)$ defines a representation in $\mathcal{F}_B(t)$ for each point $t \in T$.

In order to prove that $\rho_{(E, E_\bullet, \nabla)} \in \mathcal{F}_B(T)$, it remains to prove that $\rho_{(E, E_\bullet, \nabla)}$ defines a holomorphic map on T . It is enough to prove that it is locally holomorphic. Let $t \in T$. For each $x \in X$, there exists an open neighborhood of (x, t) , $U_{(x, t)} \subset X \times T$ over which E is trivial. Moreover, $U_{(x, t)}$ contains a product of open neighborhoods $V_{(x, t)} \subseteq X$ of x and $W_{(x, t)} \subseteq T$ of t . As for every i , the image of γ_i in X is compact, there exists a finite set of points $\{x_{i,1}, \dots, x_{i,m_i}\}$ such that $\{V_{(x_{i,j}, t)}\}_{j=1}^{m_i}$ cover γ_i . Let $W_{t,i} = \cap_{j=1}^{m_i} W_{(x_{i,j}, t)}$. Then, E is trivial over $V_{(x_{i,j}, t)} \times W_{t,i}$ for each $j = 1, \dots, m_i$, and fixing a trivialization, the connection is locally parameterized as a matrix whose entries are holomorphic functions over $W_{t,i}$.

As coordinate changes between $U_{(x_{i,j}, t)}$ are holomorphic over $W_{t,i}$, the integral $\int_\gamma \nabla_t$ is holomorphic over t and it is well defined up to the choice of a trivialization of $E|_{(x_0, t)}$. Therefore, its exponential is well defined up to conjugation and defines a holomorphic morphism over $W_{t,i}$. As this holds for every generator, and $\pi_1(X, x_0)$

is finitely generated, we obtain a holomorphic map over $W_t = \cap_{i=1}^m W_{t,i}$

$$\begin{aligned} \rho_{(E, E_\bullet, \nabla)} & : \quad \pi_1(X, x_0) \times W_t \longrightarrow \mathrm{SL}(E|_{\{(x_0, t)\}}) \\ (\gamma, t') & \longmapsto e^{-\int_\gamma \nabla_{t'}} \end{aligned}$$

The trivialization of $E|_{\{x_0\} \times T}$ over $\{x_0\} \times W_t$ allows us to identify E fibers over (x_0, t) and (x_0, t') for all $t' \in W_t$, so it induces a morphism $\rho_{(E, E_\bullet, \nabla)} : \pi_1(X, x_0) \times W_t \rightarrow \mathrm{SL}(E|_{\{x_0\} \times W_t})$ which is holomorphic in W_t .

The previous morphism is defined up to a choice of a trivialization of $E|_{\{x_0\} \times T}$ over each of the open sets W_t . By construction, changing the trivialization results in conjugating the morphism by the corresponding coordinate change $\psi : T \rightarrow \mathrm{SL}(E|_{\{x_0\} \times W_t})$. Therefore, for each (E, E_\bullet, ∇) , we obtain a well defined holomorphic morphism $\rho_{(E, E_\bullet, \nabla)} : \pi_1(X, x_0) \times W_t \rightarrow \mathrm{SL}(E|_{\{x_0\} \times W_t})$ up to conjugation by base changes $\psi : T \rightarrow \mathrm{SL}(E|_{\{x_0\} \times W_t})$.

Let us prove that the previous map define a natural transformation from \mathcal{F}_{DR} to \mathcal{F}_B , i.e., that it is functorial under base change on the parameter space T . Let S be a holomorphic variety and let $f : S \rightarrow T$ be a holomorphic map. We have to prove that the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}_{DR}(T) & \xrightarrow{\Phi(T)} & \mathcal{F}_B(T) \\ \downarrow f^* & & \downarrow f^* \\ \mathcal{F}_{DR}(S) & \xrightarrow{\Phi(S)} & \mathcal{F}_B(S) \end{array} \quad (3.8.3)$$

Let (E, E_\bullet, ∇) be a family over T in $\mathcal{F}_{DR}(T)$. We have to prove that $\rho_{(f^*E, f^*E_\bullet, f^*\nabla)} = f^*(\rho_{(E, E_\bullet, \nabla)})$. By construction, for each $s \in S$ and each $\gamma \in \pi_1(U)$,

$$\rho_{(f^*E, f^*E_\bullet, f^*\nabla)}(\gamma, s) = e^{-\int_\gamma f^*\nabla_s} \in \mathrm{SL}(f^*E|_{(x_0, s)})$$

On the other hand, if $(L, \rho) \in \mathcal{F}_B(T)$, we take $f^*(L, \rho) = (f^*L, f^*\rho)$, where

$$\begin{aligned} f^*\rho & : \quad \pi_1(U, x_0) \times S \longrightarrow \mathrm{SL}(f^*L) \\ (\gamma, s) & \longmapsto f^*\left(e^{-\int_\gamma \nabla_{f(s)}}\right) \end{aligned}$$

By definition of pullback of a vector bundle and a connection, for every $(x, s) \in X \times S$, we can identify the fiber $f^*E|_{(x, s)}$ with $E|_{(x, f(s))}$. Under this identification, if $v \in E|_{(x, f(s))}$, $(f^*\nabla_s)(f^*v) = f^*(\nabla_{f(s)}v)$. Therefore,

$$\rho_{(f^*E, f^*E_\bullet, f^*\nabla)}(\gamma, s) = e^{-\int_\gamma f^*\nabla_s} = f^*(e^{-\int_\gamma \nabla_{f(s)}}) = f^*(\rho_{(E, E_\bullet, \nabla)}(\gamma, f(s)))$$

As the latter holds for each loop γ , each point s and each (E, E_\bullet, ∇) in $\mathcal{F}_{DR}(T)$, we conclude that Φ is a natural transformation.

In regards of Lemma 3.8.4 in order to prove that the monodromy map is an isomorphism we only need to prove that it is bijective. By [Kat76], given a fixed compatible residual data, which, in our case, corresponds directly to the fixed system of weights, for every representation of the fundamental group there exists a unique holomorphic vector bundle with a logarithmic connection over the punctured curve

such that the connection has the prescribed residue and its monodromy coincides with the given one modulo conjugation.

Moreover, as the system of weights is assumed to be full flag, by Lemma 3.8.5 given such a logarithmic connection there exists a unique parabolic structure over the puncture points which is compatible with the connection. Finally, Simpson's equivalence of categories [Sim90] ensures that this correspondence is compatible with determinants and preserves parabolic stability. Therefore, for every representation of the fundamental group in $\mathcal{M}_B(X, \alpha)$ the only possible parabolic connection whose monodromy and residues are the prescribed ones must lie in $\mathcal{M}_{DR}(X, \xi, \alpha)$. Thus, the monodromy map is bijective and, therefore, it is a biholomorphism. \square

3.9 Further applications

In this section we will give some final thoughts about the framework of parabolic vector bundles described here, and describe some applications of the obtained results.

First of all, combining the results in the previous two sections, we can easily construct a parabolic analogue of the Deligne–Hitchin moduli space over a curve. In fact, obtaining a suitable construction of this moduli space was the main objective that originated our interest in the framework of parabolic Λ -modules, as we encountered the problem of the existence of the moduli space while working on Torelli type theorems for such moduli.

Let X be a smooth complex projective curve and let \overline{X} be the complex curve obtained taking the opposite complex structure on X . The Deligne–Hitchin moduli space is built by gluing together the Hodge moduli space of X and the Hodge moduli space of \overline{X} through the biholomorphisms between their respective generic fibers induced by the Riemann–Hilbert correspondence. More concretely, the Hodge moduli space of X parameterizes λ -connections over X . This space fibers over $\mathbb{A}_{\mathbb{C}}^1$ and the fiber over each $\lambda \neq 0$ is isomorphic to the moduli space $\mathcal{M}_{DR}(X, r)$ of connections on X . Similarly, the generic fiber of the moduli space of parabolic connections over \overline{X} is isomorphic to the space $\mathcal{M}_{DR}(\overline{X}, r)$ of connections over \overline{X} . On the other hand, as X and \overline{X} are homeomorphic, the moduli spaces of representations of their respective fundamental groups are canonically isomorphic. This, combined with the Riemann–Hilbert correspondence, gives us a biholomorphic identification $\mathcal{M}_{DR}(X, r) \cong \mathcal{M}_{DR}(\overline{X}, r)$, which extends to a biholomorphic identification between the generic fibers of $\mathcal{M}_{Hod}(X, r)$ and $\mathcal{M}_{Hod}(\overline{X}, r)$. The resulting moduli space is of particular interest, as it coincides with the twistor space of the moduli space of Higgs bundles.

In the parabolic case, a similar gluing construction was described in [AG16]. In this scenario the presence of an additional parameter in the form of the parabolic weights imply that there are some additional and geometrical factors that might be taken into account to ensure an adequate identification between the Hodge moduli spaces. The results presented in this work on the existence of the moduli space of parabolic λ -connections and on the regularity of the Riemann–Hilbert correspondence between the moduli space of parabolic connections and the moduli space of

representations of the fundamental group of the curve minus the parabolic points allow us to prove that the resulting space is a holomorphic variety. Moreover, we have started working towards a proof stating that the parabolic Deligne–Hitchin moduli space constructed this way coincides with the twistor space of the moduli space of strongly parabolic Higgs bundles.

In this sense, it is worth mentioning the work by Logares and Martens [LM10] describing a natural Poisson structure on the moduli space of (weakly/non-strongly) parabolic Higgs bundles. The moduli space of strongly parabolic Higgs bundles represents a symplectic leaf of this structure and gives it the structure of a hyperkähler variety. By imposing the residual structures on the space of parabolic λ -modules described through Section 3.7, we ensure that we respect this symplectic leaf both when we move through the Simpson correspondence and when we do the gluing to construct the parabolic Deligne–Hitchin.

On the other hand, I would like to make notice that the existence theorem for the parabolic Hodge moduli space proved in Section 3.7 has been applied by Gothen and Oliveira to prove the topological mirror symmetry conjecture of Hausel–Thaddeus for the moduli space of strongly parabolic Higgs bundles [GO17]. I would like to thank André Oliveira for making me notice this application of the result. In particular, he adapts the following idea from Hausel and Thaddeus [HT03]. The moduli space of parabolic λ -connections constructed in Section 3.7 is a quasi-projective variety fibering over $\mathbb{A}_{\mathbb{C}}^1$, whose fibers over $\lambda \neq 0$ are isomorphic to the moduli space of parabolic connections $\mathcal{M}_{\text{DR}}(X, r, \alpha, \xi)$ for each $\lambda \neq 0$ and whose fiber over $\lambda = 0$ is isomorphic to the moduli space of parabolic Higgs bundles $\mathcal{M}_{\text{Higgs}}(X, r, \alpha, \xi)$. Moreover, we have a canonical \mathbb{C}^* -action on the moduli space commuting with the product on $\mathbb{A}_{\mathbb{C}}^1$ and inducing an explicit isomorphism between all the non-zero fibers of the parabolic Hodge moduli space.

Therefore, we have constructed a degeneration of $\mathcal{M}_{\text{DR}}(X, r, \alpha, \xi)$ into $\mathcal{M}_{\text{Higgs}}(X, r, \alpha, \xi)$ which is given by a smooth quasi-projective variety (in fact semiprojective, c.f. [HRV15]) if the parabolic weights are full flag and generic. An argument by Hausel and Thaddeus [HT03, Sec. 6] then proves that their stringy E-polynomials (polynomials codifying their respective stringy Hodge numbers) must coincide.

Although the original argument by Hausel and Thaddeus [HT03] and the one applied by Gothen and Oliveira [GO17] were both aimed to prove that the moduli spaces of (parabolic) Higgs bundles and (parabolic) connections had the same E-polynomials, this technique could be used to compute the E-polynomials of a much broader class of moduli spaces. Given a split quasi-polynomial sheaf of rings of differential operators Λ over a variety X (c.f. [Sim94]), we can construct its reduction to the graduate Λ^R . It is a sheaf of rings of differential operators over $X \times \mathbb{A}_{\mathbb{C}}^1$ whose fiber over 1 is isomorphic to Λ and whose fiber over 0 is isomorphic to its graduate $\text{Gr}^\bullet(\Lambda)$.

If Λ is split quasi-polynomial, its graduate coincides with $\text{Sym}^\bullet(L)$ for some L , so $\text{Gr}^\bullet(\Lambda)$ -modules are just L -twisted Higgs bundles. Using Simpson construction [Sim94] in the non-parabolic case, or the construction presented in this Chapter for the parabolic scenario, we can construct the moduli spaces of (parabolic) Λ^R -modules, which fibers over $\mathbb{A}_{\mathbb{C}}^1$. Then, the fiber over a nonzero λ of this moduli space is isomorphic to the moduli space of (parabolic) Λ -modules, while the fiber over $\lambda = 0$ is isomorphic to the moduli space of L -twisted Higgs bundles. What is

more, the moduli has a natural \mathbb{C}^* -action lifting the product on $\mathbb{A}_{\mathbb{C}}^1$ and inducing isomorphisms between the nonzero fibers.

This type of moduli spaces serves to construct multiple examples of degenerations from moduli spaces of Λ -modules and parabolic Λ -modules to the moduli space of L -twisted Higgs bundles and parabolic L -twisted Higgs bundles respectively. For properly chosen sheaves Λ , using the arguments by Hausel and Thaddeus [HT03], we can reduce the computation of the E-polynomials of many types moduli spaces of Λ -modules and parabolic Λ -modules to the computation of the E-polynomials of the moduli spaces of twisted Higgs bundles and parabolic twisted Higgs bundles.

Chapter 4

Automorphism group of the moduli space of parabolic bundles over a curve

Let X be an irreducible smooth complex projective curve. Let $D = \sum_{i=1}^n x_i$ be an effective divisor on X consisting on distinct points and let ξ be a line bundle on X . Let α be a rank r generic full flag system of weights over D . Let $\mathcal{M}(r, \alpha, \xi)$ be the moduli space of stable parabolic vector bundles (E, E_\bullet) over (X, D) of rank r with system of weights α and determinant $\det(E) \cong \xi$.

Before describing the automorphisms of this moduli space, let us go back to the non-parabolic case and recall the known classification of the automorphisms of the moduli space of vector bundles. The following two transformations generate the automorphism group of the moduli space $\mathcal{M}(r, \xi)$ of stable vector bundles over X with rank r and determinant ξ . Given an automorphism $\sigma : X \rightarrow X$

1. Send $E \rightarrow X$ to $L \otimes \sigma^* E$, where L is a line bundle over X with $L^r \otimes \sigma^* \xi \cong \xi$
2. Send E to $L \otimes \sigma^*(E^\vee)$, where L is a line bundle satisfying $L^r \otimes \sigma^* \xi^{-1} \cong \xi$

This result was initially proved by Kouvidakis and Pantev [KP95] using an argument on the fibers of the Hitchin map defined on the moduli space of Higgs bundles. Hwang and Ramanan [HR04] gave a different proof based on the study of Hecke curves on the moduli space. They proved that the Hitchin discriminant was isomorphic to the union of the images of all possible Hecke curves.

Later on a simplified proof was given in [BGM13], in which the study of the Hecke transformation and the minimal rational curves on the moduli space was substituted by the geometric characterization of the nilpotent cone bundle of a generic vector bundle. This lead to the proof that given a generic bundle E whose image under the automorphism E' is itself generic, there exists an isomorphism of Lie algebra bundles

$$\mathrm{End}_0(E) \cong \mathrm{End}_0(E')$$

Then, it is proven that such an automorphism exists if and only if E' is obtained from E by one of the previously described transformations. The argument was further generalized to the moduli space of symplectic bundles in [BGM12]. In this chapter, we will generalize this result to the parabolic scenario.

Coming back to the moduli of parabolic vector bundles, first, we develop four “basic transformations” that can be applied intrinsically to families of quasi-parabolic vector bundles. The first three types come from adapting the previously mentioned ones (pullback with respect to an automorphism of the curve, tensoring with a line bundle and dualization) to parabolic vector bundles, finding naturally induced filtrations at the parabolic points on the resulting vector bundles. Nevertheless, in the parabolic setup there is a fourth new type of transformation that can be defined using the additional information provided by the parabolic structure. We can use the steps of the filtration to perform a Hecke transformation on the underlying vector bundle at the parabolic points. What is more, the full parabolic structure at each parabolic point can be “rotated” in a certain way so that it induces a parabolic structure on the resulting bundle. The possible combinations of these four types of transformations

- Taking pullback with respect to an automorphism $\sigma : X \rightarrow X$ that fixes the set of parabolic points D (but not necessarily fixes every point in D) $(E, E_\bullet) \mapsto \sigma^*(E, E_\bullet)$
- Tensoring with a line bundle $(E, E_\bullet) \mapsto (E, E_\bullet) \otimes L$
- Dualization $(E, E_\bullet) \mapsto (E, E_\bullet)^\vee$
- Hecke transformations $(E, E_\bullet) \mapsto \mathcal{H}_x(E, E_\bullet)$ with respect to the subspace $E_{x,2} \subset E|_x$ for some $x \in D$

form a group \mathcal{T} that we call group of basic transformations.

Instead of working with a fixed moduli space $\mathcal{M}(r, \alpha, \xi)$ and compute its automorphisms, it will come more natural to study the possible isomorphisms between two moduli spaces $\mathcal{M}(X, r, \alpha, \xi)$ and $\mathcal{M}(X', r', \alpha', \xi')$, leading to what is usually called an Extended Torelli type theorem. Will prove that basic transformations are the only ones giving rise to isomorphisms between moduli spaces of parabolic vector bundles. More precisely, the main result in this article is the following Theorem (see Theorem 4.6.22)

Theorem 4.0.1. *Let (X, D) and (X', D') be two smooth projective curves of genus $g \geq 6$ and $g' \geq 6$ respectively with set of marked points $D \subset X$ and $D' \subset X'$. Let ξ and ξ' be line bundles over X and X' respectively, and let α and α' be full flag generic systems of weights over (X, D) and (X', D') respectively. Let*

$$\Phi : \mathcal{M}(X, r, \alpha, \xi) \xrightarrow{\sim} \mathcal{M}(X', r', \alpha', \xi')$$

be an isomorphism. Then

1. $r = r'$
2. (X, D) is isomorphic to (X', D') , i.e., there exists an isomorphism $\sigma : X \xrightarrow{\sim} X'$ sending D to D' .
3. There exists a basic transformation T such that for every $(E, E_\bullet) \in \mathcal{M}(r, \alpha, \xi)$

$$\sigma^* \Phi(E, E_\bullet) \cong T(E, E_\bullet)$$

Apart from acting on parabolic vector bundles, the group \mathcal{T} acts on line bundles ξ and systems of weights α so that for every $T \in \mathcal{T}$, if (E, E_\bullet) has determinant ξ and is stable for the weights α , then $T(E, E_\bullet)$ has determinant $T(\xi)$ and is stable for the weights $T(\alpha)$. For T to induce an isomorphism $T : \mathcal{M}(r, \alpha, \xi) \xrightarrow{\sim} \mathcal{M}(r, \alpha, \xi')$ it is necessary and sufficient that

- $T(\xi) \cong \xi'$
- $T(\alpha)$ is in the same stability chamber as α'

This will allow us to compute the automorphism group $\text{Aut}(\mathcal{M}(r, \alpha, \xi))$ in Theorem 4.6.24.

In order to prove the theorem, we will generalize the approaches used in [BGM12] and [BGM13] to the particular features of the moduli space of parabolic vector bundles, although a deeper analysis on some invariant subspaces of the Hitchin map and the Hitchin discriminant will be necessary. We will prove that for a generic parabolic vector bundle $(E, E_\bullet) \in \mathcal{M}(r, \alpha, \xi)$ if $\sigma^*\Phi(E, E_\bullet) = (E', E'_\bullet)$ then there exists an isomorphism of Lie algebra bundles

$$\text{PEnd}_0(E, E_\bullet) \cong \text{PEnd}_0(E', E'_\bullet)$$

Using some algebraic methods, we will prove that if such isomorphism exists then (E', E'_\bullet) can be obtained from (E, E_\bullet) through the application of a basic transformation $T \in \mathcal{T}$, so for generic $(E, E_\bullet) \in \mathcal{M}(r, \alpha, \xi)$ there exists some $T \in \mathcal{T}$ such that $\sigma^*\Phi(E, E_\bullet) \cong T(E, E_\bullet)$. We will show that, in fact, there exists some constant $T \in \mathcal{T}$ such that for generic $(E, E_\bullet) \in \mathcal{M}(r, \alpha, \xi)$, $\sigma^*\Phi(E, E_\bullet) \cong T(E, E_\bullet)$ and then prove that the equality extends to the whole moduli space.

The structure of the chapter is the following. In Section 4.1 we recall the notion of parabolic vector bundle, parabolic stability and some properties of the moduli space of stable parabolic vector bundles. The precise notions of generic and concentrated systems of weights are given and we prove some technical lemmata regarding the behavior of generic parabolic vector bundles.

Parabolic Hitchin pairs and the Hitchin map are analyzed in Section 4.2. In Section 4.3 we study the geometry of the fibers of the Hitchin map corresponding to singular spectral curves, usually called the Hitchin discriminant. We prove that the image of the Hitchin discriminant can be intrinsically described from the geometry of $\mathcal{M}(r, \alpha, \xi)$ as an abstract variety. We use this description to prove a Torelli type theorem for the moduli space of parabolic vector bundles (Theorem 4.3.6).

Theorem 4.0.2. *If (X, D) and (X', D') are marked curves of genus at least 4 such that $\mathcal{M}(X, r, \alpha, \xi) \cong \mathcal{M}(X', r', \alpha', \xi')$, then $(X, D) \cong (X', D')$ and $r = r'$.*

This theorem has already been proved by Balaji, del Baño and Biswas [BdBnB01] for $r = 2$ and small parabolic weights, in the sense that parabolic stability is equivalent to stability of the underlying vector bundle. In contrast, our theorem only assumes that the parabolic weights are generic and it is valid for any rank.

Section 4.4 is devoted to describing the four kinds of “basic transformations” that can be applied intrinsically to families of quasi-parabolic vector bundles. The

parabolic version of the Hecke transformation is described and we analyze the stability of the resulting bundles. A presentation for the group \mathcal{T} of basic transformations is explicitly described and its abstract structure is computed in Proposition 4.4.9.

$$\mathcal{T} \cong \left((\mathbb{Z}^{|D|} \times \text{Pic}(X)) / \mathcal{G}_D \right) \rtimes (\text{Aut}(X, D) \times \mathbb{Z}/2\mathbb{Z})$$

where $\mathcal{G}_D < \mathbb{Z}^{|D|} \times \text{Pic}(X)$ is a (normal) subgroup isomorphic to $(r\mathbb{Z})^{|D|}$.

Then, in Section 4.5 we study the algebra of parabolic endomorphisms. Several classification and structure theorems are given. The main result of this section is the description of all the possible parabolic vector bundles which share the same Lie algebra bundle of traceless parabolic endomorphisms.

Theorem 4.0.1 is proved through Section 4.6. As a corollary, in Theorem 4.6.24 we describe the group of automorphisms of the moduli space $\mathcal{M}(r, \alpha, \xi)$ as a subgroup of the group of basic transformations \mathcal{T} described in Section 4.4, which varies depending on α and ξ . The dependence of the group on α and ξ depends on topological concerns coming from fixing the determinant ξ (arithmetic obstructions involving the rank and degree of the bundles) and an analysis of the stability chamber of α .

If we examine closely the results leading to the Extended Torelli (Theorem 4.6.22) and the computation of the automorphism group (Theorem 4.6.24) in Section 4.6, we observe a certain common underlying behavior for all moduli spaces of parabolic vector bundles. Basic transformations in \mathcal{T} induce all possible isomorphisms between moduli spaces, even crossing stability walls. Restricting ourselves to parabolic vector bundles with a fixed determinant ξ naturally imposes topological conditions on the transformations, leading to a subgroup

$$\mathcal{T}_\xi = \{T \in \mathcal{T} | T(\xi) = \xi\}$$

of transformations which preserve the determinant. Nevertheless, in general this group does not coincide with the group of automorphisms of the moduli space $\mathcal{M}(r, \alpha, \xi)$, as not all the transformations preserve α -stability. Some of them induce a wall crossing. If $g \geq 3$, wall crossings are 3-birational, in the sense that there are open subsets $\mathcal{U} \subset \mathcal{M}(r, \alpha, \xi)$ and $\mathcal{U}' \subset \mathcal{M}(r, \alpha', \xi)$ whose respective complements have codimension at least 3 such that there is an isomorphism $\mathcal{U} \cong \mathcal{U}'$. Up to this identification, basic transformations $T \in \mathcal{T}_\xi$ induce a birational transformation $T : \mathcal{M}(r, \alpha, \xi) \dashrightarrow \mathcal{M}(r, \alpha, \xi)$ which is an automorphism of some open subset whose complement has codimension at least 3. We will call this kind of maps 3-birational maps.

On the other hand, Boden and Yokogawa [BY99] proved that if α is full flag the moduli space $\mathcal{M}(r, \alpha, \xi)$ is rational, so the birational geometry of $\mathcal{M}(r, \alpha, \xi)$ is completely independent on the geometry of (X, D) apart from the dimensional level. Then, it seems like the notion of 3-birational maps (and in general k -birational maps) is more natural for the study of the moduli space of parabolic vector bundles than the analysis of the isomorphisms or general birational maps. In Section 4.7 we give a precise definition for k -birational maps and prove 3-birational versions of the Torelli theorem (4.7.5) and the Extended Torelli theorem (4.7.10). More particularly, for genus at least 4, we obtain that

Theorem 4.0.3. *If $\Phi : \mathcal{M}(X, r, \alpha, \xi) \dashrightarrow \mathcal{M}(X', r', \alpha', \xi')$ is a 3-birational map then $r = r'$ and $(X, D) \cong (X', D')$.*

Theorem 4.0.4. *If $\Phi : \mathcal{M}(X, r, \alpha, \xi) \dashrightarrow \mathcal{M}(X', r', \alpha', \xi')$ is a 3-birational map then $r = r'$ and there is an isomorphism $\sigma : (X, D) \rightarrow (X', D')$ and a basic transformation $T \in \mathcal{T}$ such that*

1. $T(\xi) = \sigma^* \xi'$
2. *For every (E, E_\bullet) for which Φ is defined, $\sigma^* \Phi(E, E_\bullet) \cong T(E, E_\bullet)$*

Then we conclude (Corollary 4.7.11) that the 3-birational automorphisms of $\mathcal{M}(r, \alpha, \xi)$ are

$$\mathrm{Aut}_{3\text{-Bir}}(\mathcal{M}(r, \alpha, \xi)) \cong \mathcal{T}_\xi < \mathcal{T}$$

Finally, we aim to describe explicitly the dependency of $\mathrm{Aut}(\mathcal{M}(r, \alpha, \xi))$ and the isomorphism class of $\mathcal{M}(r, \alpha, \xi)$ on the stability parameters α . In section 4.8 we analyze this problem for the concentrated chamber, in which α -stability is (roughly) equivalent to stability of the underlying vector bundle. We prove that in this chamber the Hecke transformation does never induce an automorphism of $\mathcal{M}(r, \alpha, \xi)$, even when combined with other basic transformations. The automorphism group is then explicitly described.

In section 4.9 we analyze the stability space Δ and the partition in stability chambers. Given two systems of weights α and β we consider the problem of determining whether all α -stable parabolic vector bundles are also β -stable or, conversely, there exists some α -stable parabolic vector bundle which is not β -stable. For the latter case to happen there must exist an α -stable parabolic vector bundle (E, E_\bullet) admitting a β -destabilizing subbundle $(F, F_\bullet) \subset (E, E_\bullet)$, in the sense that for all $(F', F'_\bullet) \subset (E, E_\bullet)$

$$\frac{\mathrm{pardeg}_\alpha(F', F'_\bullet)}{\mathrm{rk}(F')} < \frac{\mathrm{pardeg}_\alpha(E, E_\bullet)}{\mathrm{rk}(E)}$$

but

$$\frac{\mathrm{pardeg}_\beta(F, F_\bullet)}{\mathrm{rk}(F)} \geq \frac{\mathrm{pardeg}_\beta(E, E_\bullet)}{\mathrm{rk}(E)}$$

Therefore, the map

$$\begin{aligned} \Delta &\longrightarrow \mathbb{R} \\ \alpha' &\longmapsto \mathrm{rk}(F) \mathrm{pardeg}_{\alpha'}(E, E_\bullet) - \mathrm{rk}(E) \mathrm{pardeg}_{\alpha'}(F, F_\bullet) \end{aligned}$$

is negative for α and non-negative for β . Thus, if we consider the weights $\alpha_t = t\alpha + (1-t)\beta$, there must exist some $t \in (0, 1]$ such that

$$\mathrm{rk}(F) \mathrm{pardeg}_{\alpha_t}(E, E_\bullet) - \mathrm{rk}(E) \mathrm{pardeg}_{\alpha_t}(F, F_\bullet) = 0 \quad (4.0.1)$$

This equation defines a hyperplane in Δ , depending only on the numerical data for (E, E_\bullet) and (F, F_\bullet) , namely, the degrees $\deg(E)$, $\deg(F)$, the ranks $\mathrm{rk}(E)$, $\mathrm{rk}(F)$ and the parabolic type of (F, F_\bullet) , say \bar{n}_F . We call a “numerical barrier” any hyperplane on Δ obtained from an equation of the form (4.0.1) when we range over all possible choices for the integers $\deg(F)$, $\mathrm{rk}(F)$ and \bar{n}_F (the rank and degree of E are fixed in our moduli space). We say that a “numerical barrier” is “geometrical” if there actually exists some parabolic vector bundle (E, E_\bullet) and a subbundle $(F, F_\bullet) \subset (E, E_\bullet)$ with the correct invariants $\deg(F)$, $\mathrm{rk}(F)$ and \bar{n}_F . Numerical

and geometrical stability chambers are defined as the regions of Δ separated by the numerical or geometrical barriers respectively. We will also call the geometrical chambers simply stability chambers. We just proved that any two different stability chambers are separated by a numerical barrier, but it is not clear that any numerical barrier can be realized into a geometrical one, so a stability chamber can contain several numerical chambers.

We prove that there is a finite number of different chambers in Δ and we construct an invariant $\overline{M}(r, \alpha, d)$ classifying the “numerical” stability chambers in Δ . Theorem 4.9.6 proves that if the genus is big enough then the invariant $\overline{M}(r, \alpha, d)$ is in correspondence with (geometrical) stability chambers in Δ and use it to obtain a computable version of the Extended Torelli theorem 4.6.22.

Section 4.10 presents some examples, showing that the previous results are sharp in the following sense. As we proved that Hecke does not take part in any automorphism of $\mathcal{M}(r, \alpha, \xi)$ when α is concentrated, it is natural to wonder if for any of the presented basic transformations T (pullback, tensorization, dualization and Hecke) there exist a (general enough) marked curve (X, D) and a generic system of weights such that T induces an automorphism of $\mathcal{M}(r, \alpha, \xi)$. We provide an example of rank 2, 2 marked points and arbitrary genus for which the composition of Hecke with taking the pullback by some $\sigma : X \rightarrow X$ induce a nontrivial automorphism of the moduli. Moreover, dualization and tensoring induce nontrivial automorphisms, up to the usual topological constraint $T(\xi) = \xi$.

On the other hand, we can find a system of weights α of rank $r > 2$ such that the combination of Hecke and dualization induces a nontrivial involution of $\mathcal{M}(r, \alpha, \xi)$ which does not come from an involution of the curve X .

Finally, in Section 4.11 we will provide some additional comments, interpretations and applications of our work. First of all, we will put the obtained results on the classification of the isomorphisms between moduli spaces of parabolic vector bundles in the context of strong and refined Torelli type theorems, describing more deeply the classification of the isomorphism classes of moduli spaces of parabolic vector bundles in terms of their parameters (curve, rank and degree of the determinant). Then we will give some additional remarks and qualitative descriptions of the wall crossings for the moduli space of parabolic vector bundles, using the information on k -birational equivalences and the descriptions of the numerical chambers obtained through Section 4.7 and Section 4.9. We will end the chapter applying the obtained results to prove refined versions of the Torelli theorem for the moduli space of parabolic vector bundles with fixed degree.

4.1 Moduli space of parabolic vector bundles

Let X be an irreducible smooth complex projective curve. Let $D = \{x_1, \dots, x_n\}$ be a set of $n \geq 1$ different points of X and let us denote $U = X \setminus D$.

A parabolic vector bundle on (X, D) is a holomorphic vector bundle E of rank r endowed with a weighted flag on the fiber $E|_x$ over each parabolic point $x \in D$ called parabolic structure

$$\begin{aligned} E|_x &= E_{x,1} \supsetneq E_{x,2} \supsetneq \dots \supsetneq E_{x,l_x} \supsetneq E_{x,l_x+1} = 0 \\ 0 &\leq \alpha_1(x) < \alpha_2(x) < \dots < \alpha_{l_x}(x) < 1 \end{aligned}$$

We say that $\alpha_i(x)$ is the weight associated to $E_{x,i}$. We will denote by $\alpha = \{(\alpha_1(x), \dots, \alpha_{l_x}(x))\}_{x \in D}$ the system of real weights corresponding to a fixed parabolic structure. A system of weights is called full flag if $l_x = r$ for all parabolic points $x \in D$. We will use the simplified notation $(E, E_\bullet) = (E, \{E_{x,i}\})$ to denote a parabolic vector bundle.

Equivalently [Sim90], we can describe the parabolic structure as a collection of decreasing left continuous filtrations of sheaves on X , one filtration for each parabolic point. More precisely, for each $x \in D$, let $E_\alpha^x \subset E$ be a subsheaf on X indexed by a real $\alpha \geq 0$ such that

1. For every $\alpha \geq \beta$, $E_\alpha^x \subseteq E_\beta^x$
2. For every $\alpha > 0$, there exists $\varepsilon > 0$ such that $E_{\alpha-\varepsilon}^x = E_\alpha^x$
3. For every α , $E_{\alpha+1}^x = E_\alpha^x(-x)$
4. $E_0^x = E$

If E_α^x is a left continuous filtration, let $\alpha_i(x)$ be the i -th weight $\alpha \geq 0$ where the filtration jumps, i.e., such that for every $\varepsilon > 0$, $E_\alpha^x \neq E_{\alpha+\varepsilon}^x$. Then we can define the parabolic structure $\{E_{x,i}\}$ at the fiber $E|_x$ as the one having parabolic weights $\{\alpha_i(x)\}$ such that

$$E|_x / E_{x,i} \otimes \mathcal{O}_x = E / E_{\alpha_i(x)}^x$$

Reciprocally, if $\{E_{x,i}\}$ is a filtration of the fiber $E|_x$, endowed with weights $\alpha_i(x)$, define the subsheaves $E_{\alpha_i(x)}^x \subseteq E$ as the ones fitting in the short exact sequence

$$0 \longrightarrow E_{x,\alpha_i(x)} \longrightarrow E \longrightarrow E / E_{x,i} \otimes \mathcal{O}_x \longrightarrow 0$$

Then take $E_\alpha^x = E$ for $\alpha_{l_x}(x) - 1 \leq \alpha \leq \alpha_1(x)$ and $E_\alpha^x = E_{\alpha_i(x)}^x$ for $\alpha_{i-1}(x) < \alpha \leq \alpha_i(x)$. Then define E_α^x for $\alpha > \alpha_{l_x}(x)$ by the property

$$E_{\alpha+1}^x = E_\alpha^x(-x)$$

The resulting filtration E_α^x is a parabolic structure at the point x . The relations between these two formalisms will be explored further in Section 4.4. Given a parabolic vector bundle (E, E_\bullet) , we define its parabolic degree as

$$\text{pardeg}(E, E_\bullet) = \deg(E) + \sum_{x \in D} \sum_{i=1}^{l_x} \alpha_i(x) (\dim(E_{x,i}) - \dim(E_{x,i+1}))$$

As we will be working with stability conditions for different systems of weights α , it will be useful to denote

$$\text{wt}_\alpha(E, E_\bullet) = \sum_{x \in D} \sum_{i=1}^{l_x} \alpha_i(x) (\dim(E_{x,i}) - \dim(E_{x,i+1}))$$

Similarly, let

$$\text{pardeg}_\alpha(E, E_\bullet) = \deg(E) + \text{wt}_\alpha(E, E_\bullet)$$

We say that a parabolic vector bundle (E, E_\bullet) is of type $\bar{n} = (n_i(x))$ if

$$n_i(x) = \dim(E_{x,i}) - \dim(E_{x,i+1})$$

for every $i = 1, \dots, l_x$ and every $x \in D$. Then if (E, E_\bullet) is of type \bar{n} , we can write

$$\text{wt}_\alpha(E, E_\bullet) = \sum_{x \in D} \sum_{i=1}^r \alpha_i(x) n_i(x)$$

Notice that the right hand side does only depend on \bar{n} and α . We will denote it by $\text{wt}_\alpha(\bar{n})$.

Let $E' \subseteq E$ be a proper subbundle of a parabolic vector bundle (E, E_\bullet) . The parabolic structure on E induces a parabolic structure on E' as follows. For each parabolic point $x \in D$, we obtain a filtration by considering the set of subspaces $\{E'_{x,i}\} = \{E'_x \cap E_{x,j}\}$ for $j = 1, \dots, l_x$. The weight $\alpha'_i(x)$ of $E'_{x,i}$ is taken as

$$\alpha'_i(x) = \max_j \{\alpha_j(x) : F|_x \cap E_{x,j} = F_{x,i}\}$$

Then α' is a subset of the weights in α . While this would be the “canonical” form of the parabolic structure of E' , it will be useful to present it in terms of the original system of weights α . In particular, if $E' \subsetneq E$, let us take instead $\widetilde{E'_{x,i}} = E'_x \cap E_{x,i}$ for $i = 1, \dots, l_x$. Notice that while these spaces $\widetilde{E'_{x,i}}$ form a filtration of E'_x , they do not constitute a parabolic structure in the canonical sense, as there exists at least one j such that $\widetilde{E'_{x,j}} = \widetilde{E'_{x,j+1}}$. Nevertheless, we can use this other filtration to compute the parabolic degree of (E', E'_\bullet) . In particular, let us define $\bar{n}' = (n'_i(x))$ as follows

$$n'_i(x) = \dim(\widetilde{E'_{x,i}}) - \dim(\widetilde{E'_{x,i+1}}) = \dim(E'_x \cap E_{x,i}) - \dim(E'_x \cap E_{x,i+1})$$

Then $\text{wt}_{\alpha'}(E', E'_\bullet) = \text{wt}_\alpha(\bar{n}')$. If (E, E_\bullet) is full flag, then $0 \leq n'_i(x) \leq 1$ for every $i = 1, \dots, r$ and every $x \in D$. We say that a subbundle $E' \subsetneq E$ of a parabolic vector bundle is of type \bar{n}' if the induced filtration $\widetilde{E'_\bullet}$ is of type \bar{n}' .

Given parabolic vector bundles (E, E_\bullet) and (F, F_\bullet) with systems of weights α and β respectively, a morphism $\varphi : E \rightarrow F$ is called parabolic (respectively strongly parabolic) if it preserves the parabolic structure, i.e., if for every $x \in D$ and every $i = 1, \dots, l_{E,x}$ and $j = 1, \dots, l_{F,x}$ such that $\alpha_i(x) > \beta_j(x)$ (respectively $\alpha_i(x) \geq \beta_j(x)$)

$$\varphi(E_{x,i}) \subseteq F_{x,j+1}$$

We denote by $\text{PHom}((E, E_\bullet), (F, F_\bullet))$ the sheaf of local parabolic morphisms from (E, E_\bullet) to (F, F_\bullet) and write $\text{SPHom}((E, E_\bullet), (F, F_\bullet))$ for the subsheaf of strongly parabolic morphisms.

In particular, if (E, E_\bullet) is a parabolic vector bundle, an endomorphism $\varphi : E \rightarrow E$ is parabolic if for every $x \in D$ and every $i = 1, \dots, l_x$

$$\varphi(E_{x,i}) \subseteq E_{x,i}$$

We denote by $\text{PEnd}(E, E_\bullet)$ the sheaf of local parabolic endomorphisms of (E, E_\bullet) . Similarly, an endomorphism is strongly parabolic if for every $x \in D$ and every $i = 1, \dots, r$

$$\varphi(E_{x,i}) \subseteq E_{x,i+1}$$

We denote by $\text{SPEnd}(E, E_\bullet)$ the sheaf of strongly parabolic endomorphisms of (E, E_\bullet) .

The sheaves PHom and SPHom are subsheaves of the sheaf of morphisms Hom and they all coincide away from the parabolic points $D \subset X$. Following the notation in [BB05], let $T_{(E, E_\bullet), (F, F_\bullet)}$ be the torsion sheaf supported in D that fits in the following short exact sequence

$$0 \longrightarrow \text{PHom}((E, E_\bullet), (F, F_\bullet)) \longrightarrow \text{Hom}(E, F) \longrightarrow T_{(E, E_\bullet), (F, F_\bullet)} \longrightarrow 0$$

Then define $t_{(E, E_\bullet), (F, F_\bullet)}$ as the rational number such that

$$\text{rk}(E) \text{rk}(F) t_{(E, E_\bullet), (F, F_\bullet)} = \dim(T_{(E, E_\bullet), (F, F_\bullet)})$$

If $\alpha = \beta$, then $t_{E, F}$ only depends on the types \bar{n}' and \bar{n}'' of (E, E_\bullet) and (F, F_\bullet) respectively. More explicitly,

$$r' r'' t_{\bar{n}'', \bar{n}'} = \sum_{x \in D} \sum_{i > j} n''_i(x) n'_j(x)$$

Observe that if we take $\bar{n}' = \bar{n}''$, then $(r')^2 t_{\bar{n}', \bar{n}'}$ is just the dimension of the flag variety of type \bar{n}' . If L is a line bundle over X and (E, E_\bullet) is a parabolic vector bundle over (X, D) , we define the parabolic vector bundle $(E, E_\bullet) \otimes L$ as the one having underlying vector bundle $E \otimes L$ and whose filtrations are given by

$$(E \otimes L)_{x, i} = E_{x, i} \otimes L$$

This is a particular simple case of the general concept of tensor product of parabolic bundles. The general definition can be found in [Bis03].

Definition 4.1.1. *We say that a quasi-parabolic vector bundle is α -(semi)stable if for every proper subbundle $E' \subsetneq E$ with the induced parabolic structure*

$$\frac{\text{pardeg}_\alpha(E', E'_\bullet)}{\text{rk}(E')} < \frac{\text{pardeg}_\alpha(E, E_\bullet)}{\text{rk}(E)} \quad (\text{respectively } \leq) \quad (4.1.1)$$

We say that (E, E_\bullet) is α -unstable if it is not α -semistable.

Let ξ be a line bundle over X and let α be a system of weights of type \bar{n} . Let $\mathcal{M}(X, r, \alpha, \xi)$, or just $\mathcal{M}(r, \alpha, \xi)$, be the moduli space of semi-stable parabolic vector bundles (E, E_\bullet) on (X, D) of rank r with system of weights α and $\det(E) \cong \xi$. It is a complex projective scheme of dimension

$$\dim(\mathcal{M}(r, \alpha, d)) = (r^2 - 1)(g - 1) + r^2 t_{\bar{n}, \bar{n}}$$

In particular, observe that if α is full flag, i.e., if $\bar{n} = (1, \dots, 1)$, then

$$\dim(\mathcal{M}(X, r, \alpha, \xi)) = (r^2 - 1)(g - 1) + \frac{n(r^2 - r)}{2}$$

Similarly, let $\mathcal{M}(X, r, \alpha, d)$, or just $\mathcal{M}(r, \alpha, d)$ be the moduli of semistable parabolic vector bundles (E, E_\bullet) on (X, D) of rank r with system of weights α and $\deg(E) = d$. It has dimension

$$\dim(\mathcal{M}(r, \alpha, d)) = r^2(g - 1) + 1 + r^2 t_{\bar{n}, \bar{n}}$$

On the other hand, given a subbundle $E' \subsetneq E$, let us denote

$$s(E', E) = \operatorname{rk}(E') \deg(E) - \operatorname{rk}(E) \deg(E')$$

Reordering the inequality (4.1.1), we obtain that (E, E_\bullet) is α -(semi)stable if and only if for every subbundle $E' \subsetneq E$ yields

$$\begin{aligned} s(E', E) &= \operatorname{rk}(E') \deg(E) - \operatorname{rk}(E) \deg(E') \\ &> \operatorname{rk}(E) \operatorname{wt}_\alpha(E', E_\bullet) - \operatorname{rk}(E') \operatorname{wt}_\alpha(E, E_\bullet) \quad (\text{resp. } \geq) \end{aligned}$$

Moreover, if we give $E' \subsetneq E$ the induced parabolic structure from (E, E_\bullet) , there exists a unique parabolic vector bundle (E'', E'_\bullet) fitting in the short exact sequence

$$0 \longrightarrow (E', E'_\bullet) \longrightarrow (E, E_\bullet) \longrightarrow (E'', E'_\bullet) \longrightarrow 0 \quad (4.1.2)$$

in the sense that for each $\alpha \in \mathbb{R}$, the corresponding α step in each sheaf filtration form a short exact sequence and for each $\alpha > \beta$ the following diagram commutes

$$\begin{array}{ccccccc} 0 & \longrightarrow & (E')_\alpha^x & \longrightarrow & E_\alpha^x & \longrightarrow & (E'')_\alpha^x \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & (E')_\beta^x & \longrightarrow & E_\beta^x & \longrightarrow & (E'')_\beta^x \longrightarrow 0 \end{array}$$

In particular, we have

$$\begin{aligned} \deg(E) &= \deg(E') + \deg(E'') \\ \operatorname{rk}(E) &= \operatorname{rk}(E') + \operatorname{rk}(E'') \\ \operatorname{wt}_\alpha(E, E_\bullet) &= \operatorname{wt}_\alpha(E', E'_\bullet) + \operatorname{wt}_\alpha(E'', E'_\bullet) \end{aligned}$$

Therefore, (E, E_\bullet) is α -(semi)stable if and only if

$$\begin{aligned} s(E', E) &> (\operatorname{rk}(E') + \operatorname{rk}(E'')) \operatorname{wt}_\alpha(E', E'_\bullet) - \operatorname{rk}(E') (\operatorname{wt}_\alpha(E', E'_\bullet) + \operatorname{wt}_\alpha(E'', E'_\bullet)) \\ &= \operatorname{rk}(E'') \operatorname{wt}_\alpha(E', E'_\bullet) - \operatorname{rk}(E') \operatorname{wt}_\alpha(E'', E'_\bullet) \quad (\text{resp. } \geq) \end{aligned} \quad (4.1.3)$$

Then if we take (E'', E'_\bullet) fitting in the short exact sequence (4.1.2) as before, it is of type $\bar{n}'' = (n_i''(x))$, where $n_i''(x) = n_i(x) - n_i'(x)$.

Rewriting the stability condition (4.1.3) in terms of \bar{n}' and \bar{n}'' , we obtain that (E, E_\bullet) is α -(semi)stable if and only if for every \bar{n}' and for every subbundle $E' \subseteq E$ of type \bar{n}' .

$$s(E', E) > \operatorname{rk}(E'') \operatorname{wt}_\alpha(\bar{n}') - \operatorname{rk}(E') \operatorname{wt}_\alpha(\bar{n}'') \quad (\text{resp. } \geq)$$

Observe that, as $\operatorname{rk}(E') = \sum_{i=1}^r n_i'(x)$ for any $x \in D$, then the right hand side does only depend on α and \bar{n}' . Let us denote

$$s_{\min}(\alpha, \bar{n}') = r'' \operatorname{wt}_\alpha(\bar{n}') - r' \operatorname{wt}_\alpha(\bar{n}'')$$

where $r' = \sum_{i=1}^r n_i'(x)$ for any $x \in D$ and $r'' = r - r' = \sum_{i=1}^r n_i''(x)$.

Lemma 4.1.2. *Let $l > 0$ be an integer. If $g \geq 1 + \frac{l}{r-1}$ then for any system of weights α and any admissible \bar{n}' ,*

$$s_{\min}(\alpha, \bar{n}') \leq r' r'' ((g-1) + t_{\bar{n}', \bar{n}''}) - l$$

In particular, if $g \geq 3$ or $g = 2$ and $r \geq 3$

$$s_{\min}(\alpha, \bar{n}') \leq r' r'' ((g-1) + t_{\bar{n}', \bar{n}''}) - 2$$

Proof. By [BB05, Lemma 2.5.2] we have

$$s_{\min}(\alpha, \bar{n}') - r' r'' t_{\bar{n}', \bar{n}''} = r'' \text{wt}_{\alpha}(\bar{n}') - r' \text{wt}_{\alpha}(\bar{n}'') - r' r'' t_{\bar{n}', \bar{n}''} \leq 0$$

Moreover, for any $1 \leq r' < r$

$$r' r'' (g-1) - l \geq (r-1)(g-1) - l \geq 0$$

so

$$s_{\min}(\alpha, \bar{n}') - r' r'' t_{\bar{n}', \bar{n}''} \leq 0 \leq r' r'' (g-1) - l$$

□

Moving on with the stability analysis, let

$$s(\bar{n}', E) = \min_{\substack{E' \subsetneq E \\ E' \text{ of type } \bar{n}'}} s(E', E)$$

Then (E, E_{\bullet}) is α -(semi)stable if and only if, for all admissible \bar{n}'

$$s(\bar{n}', E) > s_{\min}(\alpha, \bar{n}') \quad (\text{resp. } \geq)$$

Let us denote by

$$\mathcal{M}_{\bar{n}', s}(r, \alpha, d) = \{(E, E_{\bullet}) \in \mathcal{M}(r, \alpha, d) \mid s(\bar{n}', E) = s\}$$

Lemma 4.1.3. *Let $l > 0$ be an integer. Let X be a curve of genus $g \geq 1 + \frac{l-1}{r-1}$ and let $D \subset X$ be a set of points in X . Let α and β be full flag systems of weights of rank r over (X, D) . Then the set of parabolic vector bundles $(E, E_{\bullet}) \in \mathcal{M}(r, \alpha, d)$ that are β -unstable has codimension at least l in $\mathcal{M}(r, \alpha, d)$. In particular, for $g \geq 2$ or $g = 1$ and $r \geq 3$, it has codimension at least 2 in $\mathcal{M}(r, \alpha, d)$.*

Proof. An α -stable quasi-parabolic vector bundle (E, E_{\bullet}) is β -unstable if and only if for some admissible \bar{n}' we have

$$s(\bar{n}', E) < s_{\min}(\beta, \bar{n}')$$

On the other hand, as (E, E_{\bullet}) is α -semistable, then

$$s(\bar{n}', E) \geq s_{\min}(\alpha, \bar{n}')$$

Therefore, (E, E_{\bullet}) is α -stable but β -unstable if and only if

$$(E, E_{\bullet}) \in \bigcup_{\bar{n}' \mid s_{\min}(\alpha, \bar{n}') \leq s < s_{\min}(\beta, \bar{n}')} \mathcal{M}_{\bar{n}', s}(r, \alpha, d)$$

As this is a finite union of subschemes, it is enough to prove that the complement of each component has codimension at least l . By Lemma 4.1.2, for every \bar{n}' and every $s < s_{\min}(\beta, \bar{n}')$ we have

$$s < s_{\min}(\beta, \bar{n}') \leq r' r''((g-1) + t_{\bar{n}', \bar{n}''}) - (l-1) \leq r' r''((g-1) + t_{\bar{n}', \bar{n}''})$$

Therefore, we can apply [BB05, Theorem 1.4.1] and we know that either $\mathcal{M}_{\bar{n}', s}(r, \alpha, d)$ is empty or it has codimension

$$\delta_{\bar{n}', s} = r' r''((g-1) + t_{\bar{n}', \bar{n}''}) - s \leq r' r''((g-1) + t_{\bar{n}', \bar{n}''}) - s_{\min}(\beta, \bar{n}') + 1$$

Applying again Lemma 4.1.2 we obtain that for $g \geq 1 + \frac{l-1}{r-1}$ we have $\delta_{\bar{n}', s} \geq l$. \square

Corollary 4.1.4. *Under the same hypothesis as the previous lemma, if $g \geq 1 + \frac{l-1}{r-1}$ and ξ is any line bundle over X then the set of parabolic vector bundles $(E, E_\bullet) \in \mathcal{M}(r, \alpha, \xi)$ that are β -unstable has codimension at least l in $\mathcal{M}(r, \alpha, \xi)$.*

Proof. Let $S^d \subsetneq \mathcal{M}(r, \alpha, d)$ be the subset of parabolic vector bundles (E, E_\bullet) that are α -stable but β -unstable. For each line bundle ξ of degree d , let

$$S^\xi = S^d \cap \mathcal{M}(r, \alpha, \xi)$$

Let $\xi, \xi' \in \text{Pic}^d(X)$. Then, there exists a line bundle $L \in J(X)$ such that $L^r = \xi' \otimes \xi^{-1}$. As tensoring with a line bundle preserves stability, it is clear that $(E, E_\bullet) \in S^\xi$ if and only if $(E, E_\bullet) \otimes L \in S^{\xi'}$. Therefore, $S^\xi \cong S^{\xi'}$ for every ξ and ξ' . Similarly, for every ξ and ξ' , $\mathcal{M}(r, \alpha, \xi)$ is isomorphic to $\mathcal{M}(r, \alpha, \xi')$. Therefore, we conclude that the codimension of S^ξ in $\mathcal{M}(r, \alpha, \xi)$ is the same as the codimension of S^d in $\mathcal{M}(r, \alpha, d)$, which is at least l by the previous Lemma. \square

In particular, applying the previous Corollary to $\beta = 0$ yields

Corollary 4.1.5. *Let $g \geq 1 + \frac{l-1}{r-1}$. If ξ is any line bundle over X , then the set of parabolic vector bundles $(E, E_\bullet) \in \mathcal{M}(r, \alpha, \xi)$ whose underlying vector bundle E is unstable has codimension at least l in $\mathcal{M}(r, \alpha, \xi)$. In particular, for $g \geq 2$ or $g = 1$ and $r \geq 3$ it has codimension at least 2.*

Corollary 4.1.6. *Let $g \geq 1 + \frac{l-1}{r-1}$. Let ξ be any line bundle over X and α any full flag system of weights. Let $\mathcal{M}^{\text{ss-vb}}(r, \alpha, \xi) \subset \mathcal{M}(r, \alpha, \xi)$ be the open nonempty subset parameterizing parabolic vector bundles (E, E_\bullet) whose underlying vector bundle E is semistable. Then the forgetful map*

$$p : \mathcal{M}^{\text{ss-vb}}(r, \alpha, \xi) \longrightarrow \mathcal{M}(r, \xi)$$

is dominant.

Proof. By the previous Corollary, $\mathcal{M}^{\text{ss-vb}}(r, \alpha, \xi)$ is a open subset of $\mathcal{M}(r, \alpha, \xi)$, so $\dim(\mathcal{M}^{\text{ss-vb}}(r, \alpha, \xi)) = \dim(\mathcal{M}(r, \alpha, \xi)) = \dim(\mathcal{M}(r, \xi)) + n \frac{r^2 - r}{2}$. Let S be the image of p . For every $E \in S$, the fiber $p^{-1}(E)$ is contained in the space of flags over $E|_x$ for every $x \in D$, so $\dim(p^{-1}(E)) \leq n \frac{r^2 - r}{2}$ for every $E \in \bar{S}$. Therefore

$$\dim(\mathcal{M}(r, \xi)) + n \frac{r^2 - r}{2} = \dim(\mathcal{M}^{\text{ss-vb}}(r, \alpha, \xi)) = \dim(p^{-1}(\bar{S})) \leq \dim(\bar{S}) + n \frac{r^2 - r}{2}$$

So $\dim(\bar{S}) = \dim(\mathcal{M}(r, \xi))$. As the latter, is irreducible, $\bar{S} = \mathcal{M}(r, \xi)$. \square

Now, we recall the notions of “generic” and “concentrated” systems of weights as described in [AG18b]. Given a set S and an integer k , let $\mathcal{P}^k(S)$ denote the set of subsets of size k of S . For each $0 < r' < r$, each map $I : D \rightarrow \mathcal{P}^{r'}(\{1, \dots, r\})$ and each integer $-nr^2 \leq m \leq nr^2$, let

$$A_{I,m} = \left\{ \alpha : r' \sum_{x \in D} \sum_{i=1}^r \alpha_i(x) - r \sum_{x \in D} \sum_{i \in I(x)} \alpha_i(x) = m \right\}$$

If we denote by $\mathcal{I}_{r'}$ the set of possible maps $I : D \rightarrow \mathcal{P}^{r'}(\{1, \dots, r\})$, let

$$A = \bigcup_{r'=1}^{r-1} \bigcup_{I \in \mathcal{I}_{r'}} \bigcup_{m=-nr^2}^{nr^2} A_{I,m}$$

We say that a full flag system of weights α over (X, D) is generic if $\alpha \notin A$. By [AG18b, Corollary 2.3], then there are no strictly semistable parabolic vector bundles and $\mathcal{M}(r, \alpha, \xi)$ is a smooth rational variety [BY99, Theorem 6.1].

A full flag system of weights $\alpha = \{(\alpha_1(x), \dots, \alpha_r(x))\}_{x \in D}$ is said to be concentrated if $\alpha_r(x) - \alpha_1(x) < \frac{4}{nr^2}$ for all $x \in D$. By [AG18b, Proposition 2.6], if $\deg(E)$ and $\text{rk}(E)$ are coprime and α is a full flag concentrated system of weights then for every parabolic vector bundle (E, E_\bullet) over (X, D) the following are equivalent

1. E is semistable as a vector bundle
2. E is stable as a vector bundle
3. (E, E_\bullet) is α -semistable as a parabolic vector bundle
4. (E, E_\bullet) is α -stable as a parabolic vector bundle

We introduce some extension results that will be needed later on.

Lemma 4.1.7. *Let $k > 0$. Suppose that $g \geq \frac{r(k+1)}{r-1}$ (in particular this clearly holds for any $r \geq 2$ if $g \geq 2k+2$). Let (E, E_\bullet) be a generic stable parabolic vector bundle. Then for any effective divisor F of degree k and any sheaf $\mathcal{F} \hookrightarrow \text{End}_0(E)(F)$ such that the quotient is supported on a finite set of points we have*

$$H^0(\mathcal{F}) = 0$$

Proof. By [BGM13, Lemma 2.2], there exists an open subset $\mathcal{U} \subset \mathcal{M}(r, \xi)$ such that for every $E \in \mathcal{U}$ we have $H^0(\text{End}_0(E)(F)) = 0$. By Corollary 4.1.5, for $g \geq \frac{r(k+1)}{r-1} \geq 2$, parabolic vector bundles (E, E_\bullet) whose underlying vector bundle E is semistable form a nonempty open subset of the moduli space $\mathcal{M}^{\text{ss-vb}}(r, \alpha, \xi) \subset \mathcal{M}(r, \alpha, \xi)$. Consider the preimage of \mathcal{U} by the forgetful morphism

$$p : \mathcal{M}^{\text{ss-vb}}(r, \alpha, \xi) \longrightarrow \mathcal{M}(r, \xi)$$

Therefore, for every $(E, E_\bullet) \in p^{-1}(\mathcal{U})$, $H^0(\text{End}_0(E)(F)) = 0$. Let $\mathcal{F} \subset \text{End}_0(E)(F)$ be any subsheaf whose quotient is supported on a finite set of points. Let $s \in H^0(\mathcal{F})$. As $\mathcal{F} \hookrightarrow \text{End}_0(E)(F)$, taking the image, s induces a section $\bar{s} \in H^0(\text{End}_0(E)(F))$, so we have $\bar{s} = 0$. Let $V = X \setminus \text{supp}(\text{End}_0(E)(F)/\mathcal{F})$. Then $s|_V = 0$. As $\text{End}_0(E)(F)$ is torsion free, \mathcal{F} is itself torsion free and then $s = 0$. Finally, by Corollary 4.1.6, p is dominant, so $p^{-1}(\mathcal{U})$ is an open nonempty set of $\mathcal{M}(r, \alpha, \xi)$. \square

Lemma 4.1.8. *Let M be a smooth complex scheme and let U be an open subset whose complement has codimension at least 2. Let $(\mathcal{E}, \mathcal{E}_\bullet)$ be a family of parabolic vector bundles over (X, D) parameterized by U . If $(\mathcal{E}, \mathcal{E}_\bullet)$ admits an extension to $M \times X$, then the extension is unique.*

Proof. Let $(\mathcal{F}^1, \mathcal{F}_\bullet^1)$ and $(\mathcal{F}^2, \mathcal{F}_\bullet^2)$ be families of parabolic vector bundles over (X, D) parameterized by M extending $(\mathcal{E}, \mathcal{E}_\bullet)$. Then \mathcal{F}^i are vector bundles over $M \times X$ and $\mathcal{F}_{x,j}^i$ are vector bundles over $M \times \{x\}$ extending \mathcal{E} and $\mathcal{E}_{x,j}$ respectively.

If the codimension of $M \setminus U$ in M is at least 2, then

$$\begin{aligned} \text{codim}(M \times \{x\} \setminus U \times \{x\}, M \times \{x\}) &\geq 2 \\ \text{codim}(M \times X \setminus U \times X, M \times X) &\geq 2 \end{aligned}$$

As $M \times \{x\}$ and $M \times X$ are smooth varieties, they are Serre S_2 varieties, so given a vector bundle over $U \times \{x\}$, or $U \times X$, if there exists an extension as a vector bundle to $M \times \{x\}$ or $M \times X$ respectively, then the extension is unique. Therefore, $\mathcal{F}^1 = \mathcal{F}^2$ and $\mathcal{F}_{x,j}^1 = \mathcal{F}_{x,j}^2$ for every $x \in D$ and every $j = 1, \dots, r$, so the extension of the parabolic vector bundle is unique. \square

Finally, we will briefly explain the notion of parabolic projective bundle. The filtration $E_{x,i}$ of $E|_x$ describing a parabolic structure on a vector bundle E defines a filtration by projective subspaces $\mathbb{P}(E_{x,i})$ of $\mathbb{P}(E|_x)$. Given a parabolic vector bundle (E, E_\bullet) , we define its projectivization as the projective bundle $\mathbb{P}(E)$ endowed with the following full flag of projective subspaces over each parabolic point $x \in D$

$$\mathbb{P}(E)|_x = \mathbb{P}(E_{x,1}) \supsetneq \mathbb{P}(E_{x,2}) \supsetneq \dots \supsetneq \mathbb{P}(E_{x,r})$$

In general, we define a parabolic projective bundle as a projective bundle \mathbb{P} over X endowed with a full flag of affine spaces over each parabolic point $x \in D$

$$\mathbb{P}|_x = \mathbb{P}_{x,1} \supsetneq \mathbb{P}_{x,2} \supsetneq \dots \supsetneq \mathbb{P}_{x,r}$$

Lemma 4.1.9. *Let X be a smooth complex projective curve and let D be an irreducible effective divisor over X . Then every parabolic projective bundle $(\mathbb{P}, \mathbb{P}_\bullet)$ admits a reduction to a parabolic vector bundle (E, E_\bullet)*

$$(\mathbb{P}, \mathbb{P}_\bullet) \cong (\mathbb{P}(E), \mathbb{P}(E_\bullet))$$

Moreover, if (E, E_\bullet) and (E', E'_\bullet) are any two reductions, there exists a line bundle L over X such that

$$(E', E'_\bullet) \cong (E, E_\bullet) \otimes L$$

Proof. Let P be the parabolic subgroup of $\text{GL}(r, \mathbb{C})$ consisting on upper triangular matrices. Let \mathcal{G} be the group scheme over X given by the following short exact sequence.

$$0 \rightarrow \mathcal{G} \rightarrow \text{GL}(r, \mathbb{C}) \times X \rightarrow (\text{GL}(r, \mathbb{C})/P) \otimes \mathcal{O}_D \rightarrow 0$$

Let $P\mathcal{G} = \mathcal{G}/\mathbb{C}^*$. A projective parabolic bundle is a $P\mathcal{G}$ -torsor and the reductions are reductions of structure sheaf to \mathcal{G} . From the short exact sequence

$$1 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{G} \longrightarrow P\mathcal{G} \longrightarrow 1 \tag{4.1.4}$$

we deduce that the obstruction for the existence of \mathcal{G} -reductions of a $P\mathcal{G}$ -torsor is given by $H^2(X, \mathcal{O}_X^*)$. On the other hand, \mathcal{O}_X^* fits in the exponential short exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 1$$

so there is an exact sequence

$$H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}_X) \longrightarrow H^2(X, \mathcal{O}_X^*) \longrightarrow H^3(X, \mathbb{Z})$$

As X has topological dimension 2, $H^3(X, \mathbb{Z}) = 0$. Moreover, X has complex dimension 1 and \mathcal{O}_X is coherent, so $H^2(X, \mathcal{O}_X) = 0$. Therefore $H^2(X, \mathcal{O}_X^*) = 0$ and we conclude that there the obstruction is zero. On the other hand, from sequence (4.1.4), as \mathcal{O}_X^* belong to the center of \mathcal{G} , the space of reductions of a $P\mathcal{G}$ -torsor to \mathcal{G} is a torsor for the group $H^1(X, \mathcal{O}_X^*)$. Every element in $H^1(X, \mathcal{O}_X^*)$ corresponds to a line bundle over X and it is clear that for every line bundle L

$$\mathbb{P}((E, E_\bullet) \otimes L) = \mathbb{P}(E, E_\bullet)$$

so we conclude that all the reductions are related by tensorization with a line bundle. \square

We conclude this section with a digression about (l, m) -stability for parabolic vector bundles.

Definition 4.1.10. *A parabolic vector bundle (E, E_\bullet) is (l, m) -(semi)stable if for every subbundle F with the induced parabolic structure*

$$\frac{\text{pardeg}(F, F_\bullet) + l}{\text{rk}(F)} < \frac{\text{pardeg}(E, E_\bullet) - m}{\text{rk}(E)} \quad (\text{respectively, } \leq)$$

Lemma 4.1.11. *Let $k > 0$ be an integer. Assume that*

$$g \geq m + l + 1 + \frac{l + k}{r - 1}$$

Then the (l, m) stable bundles form a nonempty Zariski open subset of $\mathcal{M}(r, \alpha, \xi)$ such that its complement has codimension at least k . In particular, for $g \geq m + 2l + 2$, then the locus of (l, m) stable bundles is nonempty for any rank.

Proof. First of all, let us prove that under that genus condition, the (l, m) stable parabolic vector bundles form a nonempty Zariski open subset of $\mathcal{M}(r, \alpha, d)$ whose complement has codimension at least k .

The proof of this first part is completely analogous to the proof of [BB05, Proposition 2.7]. For the convenience of the reader, we outline the main computations here.

Let (E, E_\bullet) be a stable parabolic vector bundle which fails to be (l, m) -stable. Then, there exists a subbundle E' of rank r' and degree d' with the induced parabolic structure and weight multiplicities $0 \leq n'_i(x) \leq 1$ such that

$$\frac{d' + \text{wt}(E', E'_\bullet) + l}{r'} \geq \frac{d + \text{wt}(E, E_\bullet) - m}{r} \quad (4.1.5)$$

Let (E'', E''_\bullet) be the parabolic vector bundle fitting in the sequence

$$0 \longrightarrow (E', E'_\bullet) \longrightarrow (E, E_\bullet) \longrightarrow (E'', E''_\bullet) \longrightarrow 0$$

Then E'' has rank $r'' = r - r'$, degree $d'' = d - d'$ and E''_\bullet is the induced parabolic filtration, which has weight multiplicities $n''_i(x) = 1 - n'_i(x)$. For simplicity in the equations, let us denote by wt , wt' and wt'' the parabolic weight $\text{wt}(E, E_\bullet)$, $\text{wt}(E', E'_\bullet)$ and $\text{wt}(E'', E''_\bullet)$ respectively. Then $\text{wt} = \text{wt}' + \text{wt}''$. Reordering the factors in equation (4.1.5) and substituting r , d and wt in terms of (E', E'_\bullet) and (E'', E''_\bullet) yields

$$(d' + d'')r' - d'(r' + r'') \leq (r' + r'') \text{wt}' - r'(\text{wt}' + \text{wt}'') + mr' + lr$$

which is equivalent to

$$d''r' - d'r'' \leq r'' \text{wt}' - r' \text{wt}'' + mr' + lr \quad (4.1.6)$$

Let $h^1 = \dim \text{PExt}^1((E'', E''_\bullet), (E', E'_\bullet)) = h^1(\text{PHom}((E'', E''_\bullet), (E', E'_\bullet)))$. As (E, E_\bullet) is stable, then $h^0 = H^0(\text{PHom}((E'', E''_\bullet), (E', E'_\bullet))) = 0$, so by Riemann-Roch formula

$$h^1 = -\chi(\text{PHom}((E'', E''_\bullet), (E', E'_\bullet))) = r'd'' - r''d' + r'r''(g - 1 + t_{\bar{n}'', \bar{n}'})$$

Applying inequality (4.1.6), we obtain

$$h^1 \leq r'' \text{wt}' - r' \text{wt}'' + mr' + lr + r'r''(g - 1 + t_{\bar{n}'', \bar{n}'})$$

Finally, let δ be the dimension of the locus of non- (l, m) -stable bundles in $\mathcal{M}(r, \alpha, d)$. Generically, if (E, E_\bullet) is not (l, m) -stable, then parabolic vector bundles (E', E'_\bullet) and (E'', E''_\bullet) constructed before can be found to be stable, so they are elements of the moduli spaces $\mathcal{M}(r', d', \alpha')$ and $\mathcal{M}(r'', d'', \alpha'')$ respectively, where α' and α'' are the systems of weights induced from α with multiplicities \bar{n}' and \bar{n}'' respectively. The possible (E, E_\bullet) fitting in the sequence

$$0 \longrightarrow (E', E'_\bullet) \longrightarrow (E, E_\bullet) \longrightarrow (E'', E''_\bullet) \longrightarrow 0$$

are then bounded by the projectivization of the parabolic Ext^1 -space, which has dimension $h^1 - 1$. Then

$$\begin{aligned} \delta &\leq \max_{\bar{n}'} \{ \dim(r', d', \alpha') + \dim(r'', d'', \alpha'') + h^1 - 1 \} \\ &= \max_{\bar{n}'} \left\{ \begin{aligned} &(r')^2(g - 1) + 1 + (r')^2 t_{\bar{n}', \bar{n}'} + (r'')^2(g - 1) + 1 + (r'')^2 t_{\bar{n}'', \bar{n}''} \\ &+ r'' \text{wt}' - r' \text{wt}'' + mr' + lr + r'r''((g - 1) + t_{\bar{n}'', \bar{n}'}) \end{aligned} \right\} \quad (4.1.7) \end{aligned}$$

From [BB05, Lemma 2.4.1], we know that

$$(r')^2 t_{\bar{n}', \bar{n}'} + (r'')^2 t_{\bar{n}'', \bar{n}''} + r'r'' t_{\bar{n}', \bar{n}''} = r^2 t_{\bar{n}, \bar{n}} - r'r'' t_{\bar{n}', \bar{n}''}$$

substituting in the previous equation yields

$$\delta \leq \max_{\bar{n}'} \left\{ \begin{aligned} &(r' + r'')^2(g - 1) - r'r''(g - 1) + 1 + r^2 t_{\bar{n}, \bar{n}} \\ &- r'r'' t_{\bar{n}', \bar{n}''} + r'' \text{wt}' - r' \text{wt}'' + mr' + lr \end{aligned} \right\}$$

By [BB05, Lemma 2.5.2], we have

$$-r'r''t_{\bar{n}',\bar{n}''} + r''\text{wt}' - r'\text{wt}'' \leq 0$$

Therefore, taking into account that $\dim(r, \alpha, d) = r^2(g-1) + 1 + r^2t_{\bar{n},\bar{n}}$ we obtain

$$\delta \leq \dim(r, \alpha, d) - \min_{r'} \{r'r''(g-1 - mr' - lr)\}$$

Then we can guarantee that $\dim \mathcal{M}(r, \alpha, d) - \delta \geq k > 0$ whenever

$$g \geq 1 + \max_{r'} \frac{mr' + lr + k}{r'r''}$$

As $r' + r'' = r$ and $r' \geq 1, r'' \geq 1$, then $\frac{1}{r'r''}$ attains its maximum value when $r' = 1$ and $r'' = r - 1$ or $r' = r - 1$ and $r'' = 1$. Simultaneously, $\frac{m}{r''}$ attains its maximum for $r'' = 1$, so the maximum of the above expression is attained at $r' = r - 1$ and $r'' = 1$, leading us to the desired bound for the genus

$$g \geq m + \frac{rl + k}{r - 1} + 1 = m + l + 1 + \frac{l + k}{r - 1}$$

Now, let $S^d \subsetneq \mathcal{M}(r, \alpha, d)$ be the subset parameterizing stable parabolic vector bundles $(E, E_\bullet) \in \mathcal{M}(r, \alpha, d)$ which are not (l, m) -stable. Notice that if $(E, E_\bullet) \in S^d$, then for every degree zero line bundle L , $(E, E_\bullet) \otimes L \in S^d$. To prove it, observe that if $(E', E'_\bullet) \subsetneq (E, E_\bullet)$ is a subbundle contradicting (l, m) -stability for (E, E_\bullet) , then $(E', E'_\bullet) \otimes L \subsetneq (E, E_\bullet) \otimes L$ contradicts (l, m) -stability for $(E, E_\bullet) \otimes L$. As the latter is always stable for any L , then S^d is invariant by tensorization with line bundles of degree 0.

For every line bundle ξ of degree d , let $S^\xi = S^d \cap \mathcal{M}(r, \alpha, \xi)$. If ξ' is another line bundle of degree d then there exists a line bundle L such that $L^\vee = \xi' \otimes \xi^{-1}$. Therefore, tensoring by L gives us an isomorphism between S^ξ and $S^{\xi'}$. Then, the fibers of the determinant map $\det : S^d \rightarrow \text{Pic}^d(X)$ are all isomorphic and, therefore, equidimensional. As the same happens with $\det : \mathcal{M}(r, \alpha, d) \rightarrow \text{Pic}^d(X)$, then we obtain that for every $\xi \in \text{Pic}^d(X)$, the codimension of S^ξ in $\mathcal{M}(r, \alpha, \xi)$ is the same as the codimension of S^d in $\mathcal{M}(r, \alpha, d)$ and the Lemma follows. \square

Lemma 4.1.12. *Let (E, E_\bullet) be a $(1, 0)$ -semistable parabolic vector bundle. Let $x \in D$ and let $1 < k \leq r$ be an integer. Let $E'_{x,k} \subsetneq E|_x$ be any subspace such that*

$$E_{x,k-1} \supsetneq E'_{x,k} \supsetneq E_{x,k+1}$$

And let E'_\bullet be the quasi-parabolic structure obtained substituting $E_{x,k}$ by $E'_{x,k}$ in E_\bullet . Then (E, E'_\bullet) is a stable parabolic vector bundle.

Proof. Let $F' \subsetneq E'$ be a subbundle. Let F'_\bullet be the parabolic structure induced by E'_\bullet on F' . We have

$$\begin{aligned} \text{wt}_x(F'_\bullet) &= \alpha_1(x) \text{rk}(F) + \sum_{i=2}^r \dim(F|_x \cap E'_{x,i})(\alpha_i(x) - \alpha_{i-1}(x)) \\ &= \alpha_1(x) \text{rk}(F) + \sum_{\substack{1 < i \leq r \\ i \neq k}} \dim(F|_x \cap E_{x,i})(\alpha_i(x) - \alpha_{i-1}(x)) + \dim(F|_x \cap E'_{x,k})(\alpha_k(x) - \alpha_{k-1}(x)) \\ &= \text{wt}_x(F_\bullet) + (\dim(F|_x \cap E'_{x,k}) - \dim(F|_x \cap E_{x,k}))(\alpha_k(x) - \alpha_{k-1}(x)) \end{aligned}$$

Every compatible $E'_{x,k}$ can be written as $E_{x,k+1} + \mathbb{C}v$ for some $v \in E_{x,k-1} \setminus E_{x,k+1}$. Therefore we have

$$\begin{aligned} \dim(F|_x \cap E'_{x,k}) &= \dim(F|_x \cap (E_{x,k+1} + \mathbb{C}v)) = \dim(F|_x) + \dim(E_{x,k+1}) + 1 - \dim(F|_x + E_{x,k+1} + \mathbb{C}v) \\ &= \dim(F|_x) + \dim(E_{x,k+1}) + 1 - \dim(F|_x + E_{x,k+1}) - 1 + \dim((F|_x + E_{x,k+1}) \cap \mathbb{C}v) \\ &= \dim(F|_x) + \dim(E_{x,k+1}) - \dim(F|_x + E_{x,k+1}) + \dim((F|_x + E_{x,k+1}) \cap \mathbb{C}v) \end{aligned}$$

As $\dim((F|_x + E_{x,k+1}) \cap \mathbb{C}v) \leq 1$, varying v the computed dimension can increase at most in 1. Therefore

$$\dim(F|_x \cap E'_{x,k}) \leq \dim(F|_x \cap E_{x,k}) + 1$$

Substituting in the weight equation yields

$$\text{wt}_x(F'_\bullet) \leq \text{wt}_x(F_\bullet) + (\alpha_k(x) - \alpha_{k-1}(x)) < \text{wt}_x(F_\bullet) + 1$$

Finally, by $(1, 0)$ semistability yields

$$\begin{aligned} \frac{\text{pardeg}(F, F'_\bullet)}{\text{rk}(F)} &= \frac{\deg(F) + \sum_{x \in D} \text{wt}_x(F'_\bullet)}{\text{rk}(F)} < \frac{\deg(F) + \sum_{x \in D} \text{wt}_x(F_\bullet) + 1}{\text{rk}(F)} \\ &\leq \frac{\text{pardeg}(E, E_\bullet)}{\text{rk}(E)} = \frac{\text{pardeg}(E, E'_\bullet)}{\text{rk}(E)} \end{aligned}$$

as this holds for every subbundle F , (E, E'_\bullet) is stable. \square

4.2 Parabolic Hitchin Pairs

Let L be a line bundle over a complex projective curve X . An L -twisted Hitchin pair over X is a pair (E, φ) consisting on a vector bundle E over X and a traceless morphism $\varphi \in H^0(\text{End}_0(E) \otimes L)$ called the field.

If L is the canonical bundle K , then a K -twisted Hitchin pair is usually known as a Higgs bundle and the morphism φ is known as the Higgs field.

Given a Hitchin pair (E, φ) , a subbundle $F \subseteq E$ is said to be φ -invariant if

$$\varphi(F) \subseteq F \otimes L$$

An L -twisted Hitchin pair is called stable (respectively semistable) if and only if for every φ -invariant proper subbundle $0 \neq F \subsetneq E$

$$\mu(F) < \mu(E) \quad (\text{respectively } \leq)$$

We will denote by $\mathcal{M}_L(r, \xi)$ the moduli space of semistable L -twisted Hitchin pairs of rank r and determinant $\det(E) \cong \xi$. Notice that by Serre duality, for $L = K$ the cotangent space of $\mathcal{M}(r, \xi)$ at a stable vector bundle E is

$$T_E^* \mathcal{M}(r, \xi) \cong H^1(\text{End}_0(E))^\vee = H^0(\text{End}_0(E) \otimes K),$$

hence, the cotangent bundle of $\mathcal{M}(r, \xi)$ lies as a subscheme of the moduli space of semi-stable Higgs bundles. In fact, it is an open subscheme.

Let us recall the definition of the Hitchin map

$$H : \mathcal{M}_L(r, \xi) \longrightarrow \mathcal{H}_L = \bigoplus_{k=2}^r H^0(X, L^k)$$

Let $S = \text{Tot}(L) = \underline{\text{Spec}} \text{Sym}^\bullet(L^{-1})$ be the total space of the vector bundle L . Let $\pi : S \rightarrow X$ be the projection and let $x \in H^0(S, \pi^*L)$ be the tautological section. Let us consider the characteristic polynomial of the field φ

$$\det(x \cdot \text{Id} - \pi^*\varphi) = x^r + \tilde{s}_1 x^{r-1} + \tilde{s}_2 x^{r-2} + \cdots + \tilde{s}_r$$

Then there exist unique sections $s_i \in H^0(X, L^i)$ such that $\tilde{s}_i = \pi^*s_i$. Note that φ is traceless by hypothesis, so $s_1 = 0$. The Hitchin map is then built sending each Hitchin pair (E, φ) to the coefficients of the characteristic polynomial of φ

$$(s_i)_{i=1}^r \in \bigoplus_{i=2}^r H^0(X, L^i)$$

The zeros of the characteristic polynomial $\det(x \cdot \text{Id} - \pi^*\varphi)$ define a curve $X_s \subset \text{Tot}(L)$ which is an r -to-1 cover of X . We call it the spectral curve at $s \in \mathcal{H}_L$.

A parabolic L -twisted Hitchin pair over a pointed curve (X, D) is a parabolic vector bundle (E, E_\bullet) over (X, D) endowed with an L -twisted strongly parabolic endomorphism $\varphi \in H^0(\text{SPEnd}_0 \otimes L)$. A $K(D)$ -twisted parabolic Hitchin pair is called a parabolic Higgs bundle.

A parabolic L -twisted Hitchin pair (E, E_\bullet, φ) is called stable (respectively semistable) if for every φ -invariant proper subbundle $0 \neq F \subsetneq E$ with the induced parabolic structure

$$\frac{\text{pardeg}(F, F_\bullet)}{\text{rk}(F)} \leq \frac{\text{pardeg}(E, E_\bullet)}{\text{rk}(E)} \quad (\text{respectively, } <)$$

We denote by $\mathcal{M}_L(r, \alpha, \xi)$ the moduli space of semistable L -twisted parabolic Hitchin pairs. Similarly to the non-parabolic case, by Serre duality if (E, E_\bullet) is a stable parabolic vector bundle

$$T_{(E, E_\bullet)}^* \mathcal{M}(r, \alpha, \xi) \cong H^1(\text{PEnd}_0(E, E_\bullet))^\vee = H^0(\text{SPEnd}_0(E, E_\bullet) \otimes K(D))$$

Therefore, the cotangent bundle of the moduli space of stable parabolic vector bundles is a subset of the moduli of parabolic Higgs bundles. In fact, it is an open subvariety.

We can define an analogue of the Hitchin map in the parabolic case by sending each parabolic Hitchin pair (E, E_\bullet, φ) to the characteristic polynomial of φ . Nevertheless, as the field is assumed to be strongly parabolic, it is nilpotent at the parabolic points, so its characteristic polynomial vanishes at D . Moreover, we require the field to be traceless, so the independent coefficient of the characteristic polynomial is always zero. Therefore, the image of the Hitchin map lies in

$$H : \mathcal{M}_L(r, \alpha, \xi) \longrightarrow \mathcal{H}'_L = \bigoplus_{i=2}^r H^0(X, L^i(-D))$$

We will be interested in computing the image of the Hitchin map both for the parabolic and non-parabolic cases. For non-parabolic Hitchin pairs, the following Lemma holds as a consequence of an argument from Beauville, Narasimhan and Ramanan [BNR89]

Lemma 4.2.1. *Let L be a line bundle over X such that $r \deg(L) > 2g$. Then the Hitchin map*

$$H : \mathcal{M}_L(r, \xi) \longrightarrow \bigoplus_{k=2}^r H^0(X, L^k)$$

is surjective.

Proof. By hypothesis $\deg(L^r) > 2g$, so L^r is very ample. Therefore, it admits a section $\tau \in H^0(X, L^r)$ with at most simple zeros. Let $\bar{\tau} = (0, 0, \dots, \tau) \in \bigoplus_{k=2}^r H^0(X, L^k)$. Then $X_{\bar{\tau}}$ has equation $x^r + \tau = 0$. As τ has at most simple zeroes, $X_{\bar{\tau}}$ is smooth. The smoothness condition for families of curves is open, so there is an open nonempty subset $U \subseteq \bigoplus_{k=2}^r H^0(X, L^k)$ such that for every $s \in U$, the spectral curve X_s is smooth.

On the other hand, from [BNR89, Proposition 3.6] there exists a bijection between torsion free sheaves of rank 1 over X_s (whose pushforward is automatically a stable pure dimension sheaf over $\text{Tot}(L)$) and stable Hitchin pairs (E, φ) over X such that $H(E, \varphi) = s$. As there always exist rank 1 torsion free sheaves over X_s , for every $s \in U$ there exists at least a stable Hitchin pair whose image by the Hitchin map is s , so

$$U \subseteq H(\mathcal{M}_L^s(r, \xi)) \subseteq \bigoplus_{k=2}^r H^0(X, L^k)$$

The set U is Zariski open and nonempty, so it is dense and H is dominant. By [Nit91, Theorem 6.1], it is also proper, so it must be surjective. \square

Let us prove the parabolic analogue for the Lemma

Lemma 4.2.2. *Suppose that $g \geq 2$ and let L be a line bundle over X such that $r \deg(L) > 2g$. Then the parabolic Hitchin map*

$$\mathcal{M}_L(r, \alpha, \xi) \longrightarrow \bigoplus_{k=2}^r H^0(X, L^k(-D))$$

is dominant.

Proof. Let (\mathcal{E}, Φ) be a versal family of traceless semistable Hitchin L -twisted pairs, where $\mathcal{E} \rightarrow \mathcal{M} \times X$ is a vector bundle and $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes p_2^* L$ satisfies that for every $t \in \mathcal{M}$, (\mathcal{E}_t, Φ_t) is a semistable Hitchin pair. By the previous corollary, the induced Hitchin map $h : \mathcal{M} \rightarrow \mathcal{H}_L$ is surjective. Let

$$\mathcal{M}' = h^{-1} \left(\bigoplus_{k=2}^r H^0(L^k(-D)) \right)$$

Then it is the closed subset of \mathcal{M} corresponding to stable pairs whose field is nilpotent at every $x \in D$. As the Hitchin map is surjective, its restriction $h : \mathcal{M}' \rightarrow \mathcal{H}'_L$ is surjective.

For every $x \in D$, let \mathcal{F}_x be the total space of the flag bundle over $\mathcal{E}|_{\mathcal{M}' \times \{x\}}$, i.e.

$$\mathcal{F}_x = \text{Tot}(\text{Fl}(\mathcal{E}|_{\mathcal{M}' \times \{x\}}))$$

Let $\pi : \mathcal{F}_x \rightarrow \mathcal{M}'$ be the projection. Taking the pullback of the versal family to \mathcal{F}_x , it is a family of triples $(\mathcal{E}, \{\mathcal{E}_{x,i}\}, \Phi)$ consisting on a vector bundle, a full flag filtration at the point x and a field. Consider the closed subset $\mathcal{H}_x \subseteq \mathcal{F}_x$ consisting on triples where the field preserves the filtration. It is closed by [Yok93, Lemma 4.3] (see [Alf17, Chapter 4] for more details). As the characteristic polynomial of $\Phi_t : E_t \rightarrow E_t \otimes L$ annihilates at x , it is nilpotent at x and therefore it admits an adapted filtration at x , $\{E_{t,x,i}\}$ such that

$$\Phi_t(E_{t,x,i}) \subseteq E_{t,x,i+1}$$

Therefore, the map $\mathcal{F}_x \rightarrow \mathcal{M}'$ is surjective. Now, let

$$\mathcal{N} = \mathcal{F}_{x_1} \times_{\mathcal{M}'} \mathcal{F}_{x_2} \times_{\mathcal{M}'} \cdots \times_{\mathcal{M}'} \mathcal{F}_{x_n} \rightarrow \mathcal{M}'$$

be the fiber product of all \mathcal{F}_x over \mathcal{M}' for $x \in D$. Taking the pullback of the families defined over \mathcal{F}_x for $x \in D$, there is a versal family over \mathcal{N} of triples $(\mathcal{E}, \mathcal{E}_\bullet, \Phi)$ such that $(\mathcal{E}, \mathcal{E}_\bullet)$ is a vector bundle over $\mathcal{N} \times X$ with a filtration over $\mathcal{N} \times D$ and $\Phi \in H^0(\text{SPEnd}_0(E, E_\bullet) \otimes p_2^* L)$ such that for every $t \in \mathcal{N}$, (\mathcal{E}_t, Φ_t) is a stable Hitchin pair.

Let $\mathcal{U} \subseteq \mathcal{N}$ be the open subset consisting on points $t \in \mathcal{N}$ such that $(\mathcal{E}_t, \mathcal{E}_{t,\bullet})$ is a stable parabolic vector bundle with respect to the parabolic weights α . By Corollary 4.1.5, there exists at least a filtered vector bundle (E, E_\bullet) such that E is stable and (E, E_\bullet) is parabolically stable. Therefore, \mathcal{U} is nonempty and thus, dense. Therefore $h(\mathcal{U}) \subseteq H(\mathcal{M}_L(r, \alpha, \xi))$ is dense in \mathcal{H}'_L . \square

Corollary 4.2.3. *Suppose that $g \geq 2$. Let U be any nonempty open subset of $\mathcal{M}(r, \alpha, \xi)$ and let L be a line bundle over X such that $r \deg(L) > 2g$. Then the linear space generated by the images of*

$$H(H^0(\text{SPEnd}_0(E, E_\bullet) \otimes L)) \subsetneq \bigoplus_{i=2}^r H^0(X, L^k(-D))$$

when (E, E_\bullet) runs over U is $\bigoplus_{i=2}^r H^0(X, L^k(-D))$.

Proof. Let $\mathcal{M}_L^{ss-vb}(r, \alpha, \xi) \subseteq \mathcal{M}_L(r, \alpha, \xi)$ be the subset of the moduli space of semi-stable parabolic Hitchin pairs consisting of pairs whose underlying parabolic vector bundle is semi-stable. Let

$$p_{(E, E_\bullet)} : \mathcal{M}_L^{ss-vb}(r, \alpha, \xi) \rightarrow \mathcal{M}(r, \alpha, \xi)$$

be the forgetful map and let $\bar{U} = p_{(E, E_\bullet)}^{-1}(U)$. Then \bar{U} is an open nonempty subset of $\mathcal{M}_L(r, \alpha, \xi)$, so it is a dense subset. The previous Lemma implies that H is dominant. Therefore, $H(\bar{U})$ is dense in $\bigoplus_{i=2}^r H^0(X, L^k(-D))$, so its linear span is the whole space. \square

In the case of Higgs bundles, i.e., when L is the canonical bundle K , a classical result by Hitchin shows that the Hitchin map becomes a complete integrable system for the moduli space of Higgs bundles. In the case of parabolic bundles, we will be interested in the following result from Faltings

Lemma 4.2.4 ([Fal93, V.(ii)]). *The parabolic Hitchin map*

$$H : \mathcal{M}_{K(D)}(r, \alpha, \xi) \longrightarrow \mathcal{H}'$$

is equidimensional.

Then, we can state some additional properties. For simplicity, let us write $\mathcal{H}' = \mathcal{H}'_{K(D)} = \bigoplus_{k=2}^r H^0(X, K^k D^{k-1})$. In order to simplify the notation, through this last part of the section let $m = \dim(\mathcal{M}(r, \alpha, \xi))$. Then $\dim(\mathcal{H}') = m$ and $\dim(\mathcal{M}_{K(D)}(r, \alpha, \xi)) = \dim(T^*\mathcal{M}(r, \alpha, \xi)) = 2m$.

Corollary 4.2.5. *Let $\mathcal{U} \subseteq \mathcal{M}(r, \alpha, \xi)$ be any nonempty open subset. Then the restriction of the parabolic Hitchin morphism to the cotangent bundle*

$$H_{\mathcal{U}} : T^*\mathcal{U} \longrightarrow \mathcal{H}'$$

is dominant.

Proof. First, observe that as $\mathcal{M}(r, \alpha, \xi)$ is irreducible [BY99], then \mathcal{U} is dense in $\mathcal{M}(r, \alpha, \xi)$ for any nonempty open subset of the moduli space. Suppose that $H_{\mathcal{U}}$ is not dominant. Let $S = \mathcal{H}' \setminus \text{Im}(H_{\mathcal{U}}) \subsetneq \mathcal{H}'$. Then $\dim(S) = m$. As $H : \mathcal{M}_{K(D)}(r, \alpha, \xi) \rightarrow \mathcal{H}'$ is equidimensional, then $\dim H^{-1}(S) = \dim(S) + m = 2m = \dim(\mathcal{M}_{K(D)}(r, \alpha, \xi))$. As $\mathcal{M}(r, \alpha, \xi)$ is irreducible, $H^{-1}(S)$ and $T^*\mathcal{U}$ are dense subsets, so they intersect. This contradicts that S does not contain any image of points in the cotangent bundle. \square

Corollary 4.2.6. *Let $\mathcal{U} \subseteq T^*\mathcal{M}(r, \alpha, \xi)$ be any open subset. Then the restriction of the parabolic Hitchin morphism to the cotangent bundle*

$$H_{\mathcal{U}} : T^*\mathcal{U} \longrightarrow \mathcal{H}'$$

is equidimensional.

Proof. Let $s \in \mathcal{H}'$. By the Lemma $\dim(H^{-1}(s)) = m$. As $H_{\mathcal{U}}^{-1}(s) \subseteq H^{-1}(s)$, then

$$\dim(H_{\mathcal{U}}^{-1}(s)) \leq \dim(H^{-1}(s)) = m$$

On the other hand, as $\dim(T^*\mathcal{M}(r, \alpha, \xi)) = 2m = \dim(\mathcal{H}') + m$. By the previous corollary $H_{\mathcal{U}}$ is dominant, so by [Har77, 3.22], for every $s \in \mathcal{H}'$, $\dim(H_{\mathcal{U}}^{-1}(s)) \geq m$. Therefore, every fiber has dimension m . \square

In particular, observe that if $s \in \mathcal{H}'$ corresponds to a smooth spectral curve X_s , then by [GL11, Lemma 3.2], if $\pi : X_s \rightarrow X$ is the covering then the fiber $H^{-1}(s)$ is isomorphic to

$$\text{Prym}(X_s/X) = \{L \in \text{Pic}(X_s) \mid \det(\pi_* L) \cong \xi\}$$

which is an irreducible abelian variety of dimension m . Then $H_U^{-1}(s)$ is dense in $H^{-1}(s)$.

In the following chapter, we will be interested in understanding how the geometry of $H_U^{-1}(s)$ relates to that of $H^{-1}(s)$ when s does not correspond to a smooth spectral curve. We will need the following proposition derived directly from the work of Faltings [Fal93]

Proposition 4.2.7. *Let $g \geq 4$. Then the complement of $T^*\mathcal{M}(r, \alpha, \xi)$ in $\mathcal{M}_{K(D)}(r, \alpha, \xi)$ has codimension at least 3.*

Proof. Combine the remark [Fal93, V.(iii)] on Theorem [Fal93, II.6.(iii)] with the codimension bound computations in [Fal93, p. 536] and Lemma [Fal93, II.7.(ii)]. Faltings proves that if $g \geq 3$ (or $g = 2$ with some additional constraints) these bounds imply that the codimension is at least 2, but the same computations prove that if $g \geq 4$ the codimension is at least 3. \square

4.3 Hitchin Discriminant and Torelli Theorem

Let $\mathcal{D} \subsetneq \mathcal{H}'$ be the divisor of the Hitchin space consisting of characteristic polynomials whose corresponding spectral curve is singular. We call $H^{-1}(\mathcal{D})$ the Hitchin discriminant. In order to simplify the notation, from now on let us write $\mathcal{H}' = W$ and let us write

$$\begin{aligned} H : \mathcal{M}_{K(D)}(r, \alpha, \xi) &\rightarrow W \\ H_0 = H_{T^*\mathcal{M}(r, \alpha, \xi)} : T^*\mathcal{M}(r, \alpha, \xi) &\longrightarrow W \end{aligned}$$

Proposition 4.3.1. *Assume that $g \geq 2$. Then the divisor \mathcal{D} has at most $n + 1$ irreducible components, which can be described as follows.*

1. *For each parabolic point $x \in D$, let \mathcal{D}_x be the set of characteristic polynomials whose spectral curve is singular over x .*
2. *Let \mathcal{D}_U be the set of characteristic polynomials whose spectral curve is singular, but it is smooth over each $x \in D$. And let $\overline{\mathcal{D}_U} \supsetneq \mathcal{D}_U$ be the set of characteristic polynomials whose spectral curve is singular over some $y \notin D$ (but not necessarily smooth over D).*

Then

$$\mathcal{D} = \overline{\mathcal{D}_U} \cup \bigcup_{x \in D} \mathcal{D}_x$$

Proof. It becomes clear that for every $s \in \mathcal{D}$, the corresponding singular curve X_s is either singular over some parabolic point $x \in D$ or it is smooth at the parabolic points $x \in D$ and it is singular over some point in $U = X \setminus D$. Therefore, $\mathcal{D} = \overline{\mathcal{D}_U} \cup \bigcup_{x \in D} \mathcal{D}_x$ and it is enough to prove that each element in the decomposition is irreducible.

Let us denote by $X_0 \subset \text{Tot}(KD)$ the image of X in the total space of KD given by the zero section of the line bundle. If a spectral curve X_s is singular over $x \in D$, it has a singular point precisely at $(x, 0) \in X_0$. A spectral curve X_s has a singularity over X_0 at $(y, 0)$ if and only if the characteristic polynomial $p_s(z, t) = t^r + \sum_{k=1}^r s_k(z)t^{r-k}$ satisfies the following properties

1. $s_r(z) \in H^0(K^r D^r)$ annihilates of order at least 2 at $z = y$, i.e., $s_r \in H^0(K^r D^r(-y))$
2. $s_{r-1}(z) \in H^0(K^{r-1} D^{r-1})$ annihilates at $z = y$, i.e., $s_{r-1} \in H^0(K^{r-1} D^{r-1}(-y))$

As $s = (s_2, \dots, s_r) \in W$, we already know that

$$s_{r-1} \in H^0(K^{r-1} D^{r-2}) \subseteq H^0(K^{r-1} D^{r-1}(-x))$$

and $s_r \in H^0(K^r D^{r-1})$, so the points in \mathcal{D}_x are precisely those with $s_r \in H^0(K^r D^{r-1}(-x))$. Therefore

$$\mathcal{D}_x = \bigoplus_{k=2}^{r-1} H^0(K^k D^{k-1}) \oplus H^0(K^r D^{r-1}(-x))$$

is irreducible for every $x \in D$.

On the other hand, let X_s be a spectral curve with a singularity over some $y \notin D$. Assume that $r > 2$. As $y \notin D$, there exists a section $t_0 \in H^0(K)$ such that the curve defined by the polynomial $p_s^{t_0}(z, t) = p_s(z, t - t_0)$ is singular at the point $(y, 0)$, but smooth over every point $x \in D$. Set $s^{t_0} = (s_i^{t_0})$ as

$$p_s^{t_0}(z, t) = (t - t_0)^r + \sum_{k>0} s_k(z)(t - t_0(z))^{r-k} = t^r + \sum_{k=1}^r s_k^{t_0}(z)t^{r-k}$$

More precisely, we have

$$s_k^{t_0} = \binom{r}{k} (-t_0)^{\otimes k} + \sum_{1 \leq j < k} \binom{r-j}{r-k} s_j \otimes (-t_0)^{\otimes k-j} \in H^0(K^k D^{k-1})$$

As $X_{s^{t_0}}$ is singular at $y \notin D$, but smooth at every $x \in D$, then applying the previous criterion yields

$$s^{t_0} \in \bigoplus_{k=1}^{r-2} H^0(K^k D^{k-1}) \oplus H^0(K^{r-1} D^{r-2}(-y)) \\ \oplus \left(H^0(K^r D^{r-1}(-2y)) \setminus \bigcup_{x \in D} H^0(K^r D^{r-1}(-2y - x)) \right) := \mathcal{R}_y$$

Observe that if $g \geq 2$ and $r > 2$ then $\deg(K^{r-1} D^{r-1}) = (r-1)(2g-2+|D|) > 3$. Therefore, for any divisor N with $0 \leq \deg(N) \leq 3$ we have

$$\deg(K^{1-r} D^{1-r}(N)) = -\deg(K^{r-1} D^{r-1}) + N < -3 + N \leq 0$$

Therefore, $h^0(K^{1-r} D^{1-r}(N)) = 0$ and, using Riemann-Roch theorem

$$h^0(K^r D^{r-1}(-N)) = \deg(K^r D^{r-1}) - N + 1 - g + h^0(K^{1-r} D^{1-r}(N)) \\ = \deg(K^r D^{r-1}) + 1 - g - N$$

Then $h^0(K^r D^{r-1}(-N)) = h^0(K^r D^{r-1}) - N$. Therefore, the last summand in the expression of \mathcal{R}_y is the complement of an hyperplane in $H^0(K^r D^{r-1}(-2y))$ so, in particular, \mathcal{R}_y is irreducible and nonempty.

Observe that as the polynomials in W have $s_1 = 0$, then $s_1^{t_0} = -rt_0$. Therefore, given any point $s' \in \mathcal{R}_y$, we can obtain a point in \mathcal{D}_U taking

$$s = (s')^{s'_1/4}$$

Therefore, for every $y \notin D$ we obtain a map from \mathcal{R}_y to \mathcal{D}_U and for every element in \mathcal{D}_U there exists an element in \mathcal{R}_y mapping to it for some $y \notin D$. Then, we can build the following variety

$$\mathcal{R} := \coprod_{x \notin D} \mathcal{R}_y \subsetneq (X \setminus D) \times W$$

Let us prove that \mathcal{R} is irreducible. Let \mathcal{R}' be the subbundle of $(X \setminus D) \times W$ whose fiber over $y \notin D$ is

$$\bigoplus_{k=1}^{r-2} H^0(K^k D^{k-1}) \oplus H^0(K^{r-1} D^{r-2}(-y)) \oplus H^0(K^r D^{r-1}(-2y)) \subsetneq W$$

Moreover, for every $x \in D$, let $\mathcal{R}^x \subsetneq \mathcal{R}'$ be the subbundle of \mathcal{R}' whose fiber over $y \notin D$ is

$$\bigoplus_{k=1}^{r-2} H^0(K^k D^{k-1}) \oplus H^0(K^{r-1} D^{r-2}(-y)) \oplus H^0(K^r D^{r-1}(-2y-x)) \subsetneq \mathcal{R}'_y$$

Then we can write

$$\mathcal{R} = \mathcal{R}' \setminus \bigcup_{x \in D} \mathcal{R}^x$$

As \mathcal{R}^x are subbundles of \mathcal{R} , then \mathcal{R} is irreducible. Finally, the maps $\mathcal{R}_y \rightarrow \mathcal{D}_U$ induce a well defined surjective map

$$\mathcal{R} \rightarrow \mathcal{D}_U$$

Therefore \mathcal{D}_U is irreducible. Moreover, from construction we obtain that $\mathcal{R}' \rightarrow \overline{\mathcal{D}_U}$ is also surjective, so $\overline{\mathcal{D}_U}$ is also irreducible.

It remains to consider the case $r = 2$, but in that case we have simply $W = H^0(K^2 D)$. Then the spectral curve corresponding to a point $s = s_2 \in W$ has equation $t^2 + s_2(z) = 0$. Therefore, it has a singularity over $y \notin D$ if and only if s_2 annihilates of order at least 2 in y , i.e., for $y \in H^0(K^2 D(-2y))$. Then

$$\mathcal{D} = \bigcup_{y \notin D} H^0(K^2 D(-2y)) \cup \bigcup_{x \in D} H^0(K^2 D(-x))$$

As $g \geq 2$, the first union is the image of the subbundle $\mathcal{R}' \hookrightarrow U \times W$ whose fiber over $y \in U$ is $H^0(K^2 D(-2y))$ under the projection map

$$\mathcal{R}' \hookrightarrow U \times W \xrightarrow{p_W} W$$

so it is irreducible and corresponds to $\overline{\mathcal{D}_U}$. □

Proposition 4.3.2. *Suppose that $g \geq 4$. Then for $s \in W \setminus \mathcal{D}$ the fiber $H_0^{-1}(s)$ is an open subset of an abelian variety. For a generic s in each irreducible component of \mathcal{D} the fiber $H_0^{-1}(s)$ contains a complete rational curve.*

Proof. By [GL11, Lemma 3.2], if X_s is smooth and $\pi : X_s \rightarrow X$ is the covering then the fiber $H^{-1}(s)$ is isomorphic to

$$\mathrm{Prym}(X_s/X) = \{L \in \mathrm{Pic}(X_s) \mid \det(\pi_* L) \cong \xi\}$$

which is an abelian variety.

On the other hand, if $s \in \mathcal{D}_U$ is generic then X_s has a unique singularity which is a node not lying over a parabolic point. By [Bho96, Theorem 4] the fiber $H^{-1}(s)$ is an uniruled variety. More precisely, it is birational to a \mathbb{P}^1 -fibration over the Jacobian $J(\tilde{X}_s)$, where, \tilde{X}_s is the normalization of X_s .

Let $Z = (\mathcal{M}_{K(D)}(r, \alpha, \xi) \setminus T^*\mathcal{M}(r, \alpha, \xi)) \cap H^{-1}(\mathcal{D}_U)$. By Proposition 4.2.7, for $g \geq 4$, the complement $\mathcal{M}_{K(D)}(r, \alpha, \xi) \setminus T^*\mathcal{M}(r, \alpha, \xi)$ has codimension at least 3. Therefore, Z has codimension at least 2 in $H^{-1}(\mathcal{D}_U)$. Let $S = H(\mathcal{M}_{K(D)} \setminus T^*\mathcal{M}(r, \alpha, \xi))$. Let $m = \dim \mathcal{H}'$ and assume that $\dim(S) < m - 1$. Then for any $s \in \mathcal{D}_U \setminus S$ we have $H_0^{-1}(s) = H^{-1}(s)$ and the fiber contains a complete rational curve.

Now, let us suppose that $\dim(S) = m - 1$. Then $Z \rightarrow \mathcal{D}_U$ is dominant and, therefore, the generic fiber has dimension $\dim(Z) - \dim(\mathcal{D}_U) \leq m - 2$. In other words, for a generic $s \in \mathcal{D}_U$, $Z \cap H^{-1}(s)$ has codimension at least 2 in $H^{-1}(s)$. As the latter is uniruled and we are only taking away a codimension 2 set, then there exists at least a complete rational curve in $H_0^{-1}(s)$.

It is only left to prove that a generic fiber over \mathcal{D}_x contains a complete rational curve. As $g \geq 4$, let $\mathcal{U} \subsetneq \mathcal{M}(r, \alpha, \xi)$ be the intersection of the open nonempty subsets defined by Lemma 4.1.7 and Lemma 4.1.11 for $(l, m) = (1, 0)$. It parameterizes $(1, 0)$ stable parabolic vector bundles (E, E_\bullet) such that $H^0(\mathrm{PEnd}_0(E, E_\bullet)(x_0)) = 0$.

Then for every $(E, E_\bullet) \in \mathcal{U}$ and every $x \in D$ we have

$$H^1(\mathrm{SPend}_0(E, E_\bullet) \otimes K(D - x)) = H^0(\mathrm{SPend}_0(E, E_\bullet)(x))^\vee = 0$$

so the evaluation morphism

$$\mathrm{ev} : H^0(\mathrm{SPend}_0(E, E_\bullet) \otimes K(D)) \rightarrow \mathrm{SPend}_0(E, E_\bullet) \otimes K(D)|_x$$

is surjective.

For $1 < k \leq r$, let $N_k(E, E_\bullet) \subsetneq \mathrm{SPend}_0(E, E_\bullet) \otimes K(D)|_x$ be the subspace of matrices with a zero in position $(k - 1, k)$. For $k = 1$, let $N_i(E, E_\bullet)$ be the subspace of matrices with a zero in position $(r, 1)$. Let $\tilde{N}_k(E, E_\bullet)$ be the preimage of N_i under the evaluation map. For $k \neq 1$, we can describe $\tilde{N}_k(E, E_\bullet)$ as follows. Let E_\bullet^k be the subfiltration of E obtained removing the element $E_{x,k}$. Then

$$\tilde{N}_k(E, E_\bullet) = H^0(\mathrm{SPend}_0(E, E_\bullet^k) \otimes K(D))$$

Let $(E, E_\bullet, \varphi) \in H^{-1}(\mathcal{D}_x) \cap T^*\mathcal{U}$. Let z be a coordinate on X around the parabolic point $x \in D$. Then, locally, φ can be written as

$$\varphi(z) = \begin{pmatrix} za_{11} & a_{12} & \cdots & a_{1r} \\ za_{21} & za_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ za_{r1} & za_{r2} & \cdots & za_{rr} \end{pmatrix}$$

Where a_{ij} are local sections of $K(D)$ and φ is expressed in a basis which is adapted to the parabolic filtration. Then $(E, E_\bullet, \varphi) \in H^{-1}(\mathcal{D}_x)$ if and only if $z^2 | \det(\varphi(z))$. Nevertheless, if we express the determinant as a sum of products of elements of the matrix above it becomes clear that the only summand that is not a multiple of z^2 is precisely $za_{r1}a_{12}a_{23} \cdots a_{r-1,r}$. Therefore, the determinant is a multiple of z^2 if and only if at least one of the elements a_{r1} , or $a_{k-1,k}$ annihilates at $z = 0$ for some $k > 1$. This is equivalent to ask $\text{ev}(\varphi) \in N_k$ for some $1 \leq k \leq r$. As the evaluation map is surjective for every parabolic vector bundle in \mathcal{U} , we conclude that for every $(E, E_\bullet) \in \mathcal{U}$

$$H^{-1}(\mathcal{D}_x) \cap T_{(E, E_\bullet)}^* \mathcal{M}(r, \alpha, \xi) = \bigcup_{k=1}^r \tilde{N}_k(E, E_\bullet)$$

Let $\tilde{N}_k = \bigcup_{(E, E_\bullet) \in \mathcal{U}} \tilde{N}_k(E, E_\bullet)$. By construction $\dim(\tilde{N}_k) = \dim(2\mathcal{M}(r, \alpha, \xi) - 1)$. Assume that $(E, E_\bullet, \varphi) \in \tilde{N}_k$ for some $k > 1$. By Lemma 4.1.12, for every $(E, E_\bullet) \in \mathcal{U}$, every $x \in D$, every $1 < k \leq r$ and every $E'_{x,k}$ such that $E_{x,k-1} \supsetneq E'_{x,k} \supsetneq E_{x,k+1}$ then (E, E'_\bullet) is a stable parabolic vector bundle. Moreover, as φ sends $E_{x,k-1}$ to $E_{x,k+1}$, then $\varphi \in H^0(\text{SPEnd}_0(E, E'_\bullet) \otimes K(D))$ for every choice of $E'_{x,k}$. Therefore, for every $E'_{x,k}$, $(E, E'_\bullet, \varphi) \in H_0^{-1}(\mathcal{D}_x)$. As E and φ do not change, all those Higgs bundles lie over the same point of the Hitchin map. The space of possible compatible maps in the filtration is parameterized by \mathbb{P}^1 , so they form a complete rational curve in $T^*\mathcal{M}(r, \alpha, \xi)$.

Therefore, the image of the complete rational curves contains $H(\tilde{N}_k) \subseteq \mathcal{D}_x$ for every $k > 1$. Then it is enough to prove that the image is dense for some $k > 1$. Assume that $H(\tilde{N}_k)$ is not dense. Let $S = H(\tilde{N}_k)$ and $m = \dim(\mathcal{M}(r, \alpha, \xi))$. Then $\dim(S) < \dim(\mathcal{D}_x) = m - 1$. By equidimensionality of H_0 , $\dim H_0^{-1}(S) = m + \dim(S) < 2m - 1 = \dim(\tilde{N}_k)$, but $\tilde{N}_k \subseteq H_0^{-1}(S)$. \square

Lemma 4.3.3. *Let $\mathcal{R} \subset T^*\mathcal{M}(r, \alpha, \xi)$ be the union of the complete rational curves in $T^*\mathcal{M}(r, \alpha, \xi)$. Then \mathcal{D} is the closure of $H_0(\mathcal{R})$ in W .*

Proof. Let $\mathbb{P}^1 \hookrightarrow T^*\mathcal{M}(r, \alpha, \xi)$ be a complete rational curve. Composing with the Hitchin map, we obtain a morphism

$$\mathbb{P}^1 \hookrightarrow T^*\mathcal{M}(r, \alpha, \xi) \longrightarrow W$$

from \mathbb{P}^1 to an affine space, so it is constant. Therefore, each complete rational curve must be contained in a fiber of the Hitchin morphism.

Let $s \in W \setminus \mathcal{D}$. By the previous Proposition 4.3.2, $H_0^{-1}(s)$ is an open subset of an abelian variety, so it does not admit any nonconstant morphism from \mathbb{P}^1 . Therefore, there is no complete rational curve over $W \setminus \mathcal{D}$. Then, by the second part of the Proposition 4.3.2, we know that for every irreducible component of \mathcal{D} , a generic fiber contains a rational curve. Therefore, $H_0(\mathcal{R})$ is dense in \mathcal{D} and, as \mathcal{D} is closed in W , $\mathcal{D} = \overline{\mathcal{R}}$. \square

Proposition 4.3.4. *The global algebraic functions $\Gamma(T^*\mathcal{M}(r, \alpha, \xi))$ produce a map*

$$\tilde{h} : T^*\mathcal{M}(r, \alpha, \xi) \longrightarrow \text{Spec}(\Gamma(T^*\mathcal{M}(r, \alpha, \xi))) \cong W \cong \mathbb{C}^m$$

which is the parabolic Hitchin map up to an isomorphism of \mathbb{C}^m , where $m = \dim W$. Moreover, consider the action of \mathbb{C}^* on $T^*\mathcal{M}(r, \alpha, \xi)$ given by dilatation on the fibers. Then there is a unique \mathbb{C}^* action on W such that \tilde{h} is \mathbb{C}^* -equivariant, i.e., such that

$$\tilde{h}(E, E_\bullet, \lambda\varphi) = \lambda \cdot \tilde{h}(E, E_\bullet, \varphi)$$

Proof. The Hitchin map

$$H : \mathcal{M}_{K(D)}(r, \alpha, \xi) \longrightarrow W$$

is projective and has connected fibers (see, for example, [AG18b, Lemma 3.1 and Lemma 3.2]). Then each holomorphic function $f : \mathcal{M}_{K(D)}(r, \alpha, \xi) \longrightarrow \mathbb{C}$ factors through W and, as W is affine, we obtain that

$$\mathrm{Spec}(\Gamma(\mathcal{M}_{K(D)}(r, \alpha, \xi))) \cong \mathrm{Spec}(\Gamma(W)) \cong W$$

Let $f : T^*\mathcal{M}(r, \alpha, \xi) \longrightarrow \mathbb{C}$. By [Fal93, V.(iii)], we know that the codimension of the complement of $T^*\mathcal{M}(r, \alpha, \xi)$ in $\mathcal{M}_{K(D)}(r, \alpha, \xi)$ is at least 2. As α is generic, $\mathcal{M}_{K(D)}(r, \alpha, \xi)$ is smooth. Therefore, by Hartog's theorem f extends to a holomorphic function $\bar{f} : \mathcal{M}_{K(D)}(r, \alpha, \xi) \longrightarrow \mathbb{C}$, so we conclude that $\Gamma(T^*\mathcal{M}(r, \alpha, \xi)) = \Gamma(\mathcal{M}_{K(D)}(r, \alpha, \xi))$. Therefore, we obtain a map

$$\tilde{h} : T^*\mathcal{M}(r, \alpha, \xi) \longrightarrow \mathrm{Spec}(\Gamma(T^*\mathcal{M}(r, \alpha, \xi))) \cong W$$

The \mathbb{C}^* action on the cotangent bundle then descends to a unique action on $\mathrm{Spec}(\Gamma(T^*\mathcal{M}(r, \alpha, \xi)))$ making \tilde{h} a \mathbb{C}^* -equivariant map. \square

The previous Lemma allow us to recover the Hitchin map canonically with the corresponding \mathbb{C}^* action up to an automorphism of the Hitchin space. The \mathbb{C}^* action stratifies the space W in subspaces corresponding to the elements whose rate of decay is at least $|\lambda|^k$ for each $k = 2, \dots, r$. Observe that, at first, the \mathbb{C}^* action only allows us to recover a filtration of W , but, in particular, we can recover uniquely the subspace of maximal decay $|\lambda|^r$, which corresponds to

$$W_r = H^0(K^r D^{r-1}) \subsetneq W$$

In general, for $k > 1$, let $W_k = H^0(K^k D^{k-1})$. Let

$$h_k : H^0(\mathrm{SPEnd}_0(E) \otimes K_X(D)) \rightarrow W_k$$

be the composition of the Hitchin map $H : H^0(\mathrm{SPEnd}_0(E) \otimes K_X(D)) \rightarrow W$ with the projection $W \twoheadrightarrow W_k$.

Proposition 4.3.5. *The intersection $\mathcal{C} := \mathcal{D} \cap W_r \subsetneq W_r$ has $n + 1$ irreducible components*

$$\mathcal{C} = \mathcal{C}_X \cup \bigcup_{x \in D} \mathcal{C}_x$$

As $r \geq 2$, the linear series $|K^r D^{r-1}|$ is very ample and induces an embedding $X \subset \mathbb{P}(W_r^*)$. Then $\mathbb{P}(\mathcal{C}_X) \subset \mathbb{P}(W_r)$ is the dual variety of $X \subset \mathbb{P}(W_r^*)$ and for each $x \in D$, $\mathbb{P}(\mathcal{C}_x) \subset \mathbb{P}(W_r)$ is the dual variety of $x \hookrightarrow X \subset \mathbb{P}(W_r^*)$.

Proof. A spectral curve X_s corresponding to a point $s = s_r \in H^0(K^r D^{r-1})$ has equation $t^r + s_r(z) = 0$. Therefore, it is singular precisely at the points $(z, t) = (x, 0)$ where x is a zero of order at least 2 of s_r . Observe that the equation is an equation on the points of the total space of KD so, as in previous lemmata, here we are considering s_r as a section of $K^r D^r$. Therefore, $s \in \mathcal{C}$ if and only if $s_r \in H^0(K^r D^r(-2x))$ for some $x \in X$. As we already know that $s_r \in H^0(K^r D^{r-1})$ we have two possible cases

1. $s_r \in H^0(K^r D^{r-1}(-2x))$ for some $x \notin D$
2. $s_r \in H^0(K^r D^{r-1}(-x))$ for some $x \in D$

Therefore, we can write

$$\begin{aligned} \mathcal{C} &= \bigcup_{x \in U} H^0(K^r D^{r-1}(-2x)) \cup \bigcup_{x \in D} H^0(K^r D^{r-1}(-x)) \\ &= \bigcup_{x \in X} H^0(K^r D^{r-1}(-2x)) \cup \bigcup_{x \in D} H^0(K^r D^{r-1}(-x)) \end{aligned}$$

Let us denote

$$\begin{aligned} \mathcal{C}_X &= \bigcup_{x \in X} H^0(K^r D^{r-1}(-2x)) \\ \mathcal{C}_x &= H^0(K^r D^{r-1}(-x)) \quad x \in D \end{aligned}$$

In the proof of Proposition 4.3.1 we already proved that \mathcal{C}_X and, obviously, \mathcal{C}_x are irreducible for every $x \in D$. Moreover, for $g \geq 2$, the Riemann-Roch computations carried out in the proof of Proposition 4.3.1 imply that \mathcal{C}_X and \mathcal{C}_x have both codimension 1 in W_r and they are distinct, so they are precisely the irreducible components of \mathcal{D} .

For the second part of the Proposition, observe that if $X \subseteq \mathbb{P}(W_r^*)$ is the embedding given by the linear system $|K^r D^{r-1}|$, then the set of hyperplanes in $\mathbb{P}(W_r^*)$ which are tangent to X at $x \in X$ is precisely $\mathbb{P}(H^0(K^r D^{r-1}(-2x)))$. Therefore, the dual variety of X is $\mathbb{P}(\mathcal{C}_X) \subset \mathbb{P}(W_r)$. Furthermore, for every $x \in D$, the dual variety of $x \subset \mathbb{P}(W_r^*)$ identifies with the set of hyperplanes passing through x , which is precisely $\mathbb{P}(H^0(K^r D^{r-1}(-x)))$. Therefore, we conclude that the dual of $x \subset \mathbb{P}(W_r^*)$ is $\mathbb{P}(\mathcal{C}_x) \subset \mathbb{P}(W_r)$. \square

Notice that for every $x \in D$, $\mathbb{P}(\mathcal{C}_x)$ is the dual variety of a point and $\mathbb{P}(\mathcal{C}_X)$ is the dual variety of a compact Riemann surface so, $\mathbb{P}(\mathcal{C}_X) \not\cong \mathbb{P}(\mathcal{C}_x)$ for all $x \in X$. For every $x \in D$, $\mathcal{C}_x \subset W_r$ is an hyperplane. In particular, this allows us to identify canonically \mathcal{C}_X inside \mathcal{C} as the only irreducible component that is not an an hyperplane in W_r .

Theorem 4.3.6 (Torelli theorem). *Let (X, D) and (X', D') be two smooth projective curves of genus $g \geq 4$ and $g' \geq 4$ respectively with set of marked points $D \subset X$ and $D' \subset X'$. Let ξ and ξ' be line bundles over X and X' respectively, and let α and α' be full flag generic systems of weights over (X, D) and (X', D') respectively. Then if $\mathcal{M}(X, r, \alpha, \xi)$ is isomorphic to $\mathcal{M}(X', r', \alpha', \xi')$ then $r = r'$ and (X, D) is isomorphic to (X', D') , i.e., there exists an isomorphism $X \cong X'$ sending D to D' .*

Proof. In order to simplify the notation, let $\mathcal{M} = \mathcal{M}(X, r, \alpha, \xi)$ and $\mathcal{M}' = \mathcal{M}(X', r', \alpha', \xi')$. First, let us prove that $r = r'$, $g = g'$ and $|D| = |D'|$. If \mathcal{M} and \mathcal{M}' are isomorphic, then they have the same dimension, so

$$\begin{aligned} (r-1) \left[(r-1)(g-1) + \frac{|D|}{2}r \right] &= \dim(\mathcal{M}) \\ &= \dim(\mathcal{M}') = (r'-1) \left[(r'-1)(g'-1) + \frac{|D'|}{2}r' \right] \end{aligned} \quad (4.3.1)$$

On the other hand, if $\Phi : \mathcal{M} \xrightarrow{\sim} \mathcal{M}'$ is an isomorphism, then there is an isomorphism $d(\Phi^{-1}) : T^*\mathcal{M} \xrightarrow{\sim} T^*\mathcal{M}'$ which is \mathbb{C}^* equivariant for the standard dilatation action. By Proposition 4.3.4, there exist unique \mathbb{C}^* actions \cdot and \cdot' on $\Gamma(T^*\mathcal{M})$ and $\Gamma(T^*\mathcal{M}')$ respectively that are compatible with the dilatation on the fibers. Therefore, there must exist an algebraic \mathbb{C}^* -equivariant isomorphism $f : \Gamma(T^*\mathcal{M}) \xrightarrow{\sim} \Gamma(T^*\mathcal{M}')$ such that the following diagram commutes

$$\begin{array}{ccc} T^*\mathcal{M} & \xrightarrow{d(\Phi^{-1})} & T^*\mathcal{M}' \\ \tilde{h} \downarrow & & \downarrow \tilde{h} \\ \Gamma(T^*\mathcal{M}) & \xrightarrow{f} & \Gamma(T^*\mathcal{M}') \end{array}$$

As f is \mathbb{C}^* -equivariant, it must preserve the filtration by subspaces in terms of the decay and it must send the subspace of maximum decay $|\lambda|^r$ of $\Gamma(T^*\mathcal{M})$ to the subspace of maximum decay $|\lambda|^{r'}$ of $\Gamma(T^*\mathcal{M}')$. Therefore, the number of steps of the filtration must be the same and the spaces of top decay must have the same dimension. As the filtrations of $\Gamma(T^*\mathcal{M})$ and $\Gamma(T^*\mathcal{M}')$ have $r-1$ and $r'-1$ steps respectively, then $r = r'$. The dimension of such subspaces are the dimensions of $W_r = H^0(K_X^r D^{r-1})$ and $W'_r = H^0(K_{X'}^r (D')^{r-1})$ respectively, so

$$(r-1)(2g-2+|D|) = h^0(K_X^r D^{r-1}) = h^0(K_{X'}^r (D')^{r-1}) = (r-1)(2g'-2+|D'|)$$

But then, multiplying the equality by $(r-1)/2$ and subtracting it to the equation (4.3.1) yields

$$\frac{r}{2}|D| = \frac{r}{2}|D'|$$

So $|D| = |D'|$. Now substituting in the dimension of W_r and W'_r we obtain $g = g'$ as desired.

By Proposition 4.3.4, there is an isomorphism $W \cong W'$, that we will denote by a slight abuse of notation as $f : W \rightarrow W'$, such that the following diagram commutes

$$\begin{array}{ccc} T^*\mathcal{M} & \xrightarrow{d(\Phi^{-1})} & T^*\mathcal{M}' \\ H_0 \downarrow & & \downarrow H'_0 \\ W & \xrightarrow{f} & W' \end{array}$$

and there exist unique \mathbb{C}^* actions on W and W' such that H_0 and H'_0 are \mathbb{C}^* -equivariant. As $d(\Phi^{-1})$ is an isomorphism, it maps complete rational curves on $T^*\mathcal{M}$ to complete rational curves on $T^*\mathcal{M}'$. By Lemma 4.3.3, f sends the locus of

singular spectral curves $\mathcal{D} \subset W$ to the locus of singular spectral curves $\mathcal{D}' \subset W'$. On the other hand, the differential map $d(\Phi^{-1})$ is \mathbb{C}^* -equivariant, so f must be a \mathbb{C}^* -equivariant map. Therefore, it must send the subspace of W of elements with maximum decay W_r to the subspace of W' of elements with maximum decay W'_r . Let $f_r : W_r \rightarrow W'_r$ be the restriction of f to W_r .

By definition of W_r , we know that f_r is \mathbb{C}^* -equivariant and homogeneous of degree r , so it must be linear, and it maps $\mathcal{C} = \mathcal{D} \cap W_r$ to $\mathcal{C}' = \mathcal{D}' \cap W'_r$. Let \mathcal{C}_X and \mathcal{C}'_X be the only components of \mathcal{C} and \mathcal{C}' respectively that are not hyperplanes. By Proposition 4.3.5, the dual variety of $\mathbb{P}(\mathcal{C}_X)$ in $\mathbb{P}(W_r)$ is $X \subset \mathbb{P}(W_r^*)$ and, similarly, the dual variety of $\mathbb{P}(\mathcal{C}'_X)$ in $\mathbb{P}(W'_r)$ is $X' \subset \mathbb{P}((W'_r)^*)$, so f induces an isomorphism $f^\vee : \mathbb{P}(W_r^*) \rightarrow \mathbb{P}((W'_r)^*)$ that sends X to X' . Moreover, the dual of the rest of the components $\mathbb{P}(\mathcal{C}_x)$ of $\mathbb{P}(\mathcal{C})$ correspond to the divisor $D \subset X \subset \mathbb{P}(W_r^*)$ and the dual of the components $\mathbb{P}(\mathcal{C}'_x)$ of $\mathbb{P}(\mathcal{C}')$ correspond to the divisor $D' \subset X' \subset \mathbb{P}((W'_r)^*)$, so f^\vee must send D to D' . Therefore, f^\vee induces an isomorphism $f^\vee : (X, D) \xrightarrow{\sim} (X', D')$. \square

4.4 Basic transformations for quasi-parabolic vector bundles

Let $x \in X$ be a point. Given a vector bundle E over X and a subspace on the fiber $H \subseteq E|_x$, the Hecke transformation of E at x with respect to the subspace H is defined as the subsheaf $\mathcal{H}_x^H(E) \subseteq E$ fitting in the short exact sequence

$$0 \longrightarrow \mathcal{H}_x^H(E) \longrightarrow E \longrightarrow (E|_x/H) \otimes \mathcal{O}_x \longrightarrow 0$$

this kind of transformations were first studied in [NR778, HR04] and have been used broadly to study the geometry of the moduli spaces of vector bundles. Let $x \in D$ be a parabolic point. For each parabolic vector bundle (E, E_\bullet) on (X, D) , each term in the parabolic filtration $E_{x,i} \subseteq E|_x$ for $1 \leq i \leq l_x + 1$ gives us a canonical choice for a linear subspace in the fiber $E|_x$, so we might define subsheaves $E_x^i \subseteq E$ through the Hecke transformation as

$$0 \longrightarrow E_x^i \longrightarrow E \longrightarrow (E|_x/E_{x,i}) \otimes \mathcal{O}_x \longrightarrow 0 \quad (4.4.1)$$

Note that for each $x \in D$ and each $i = 1, \dots, l_x + 1$, these subsheaves E_x^i coincide with the jumps at the continuous parabolic filtration associated to (E, E_\bullet)

$$E_x^i = E_{\alpha_i(x)}^x$$

In fact, the Hecke transformation gives us a one to one correspondence between parabolic structures $\{E_{x,i}\}$ on E and collections of decreasing sequences of subsheaves

$$E = E_x^1 \supsetneq E_x^2 \supsetneq \dots \supsetneq E_x^{l_x} \supsetneq E_x^{l_x+1} = E(-x)$$

for every $x \in D$.

Let us restrict the short exact sequence (4.4.1) to the point x . If $f : E_x^i|_x \rightarrow E|_x$ is the induced map at the fiber, we get

$$\begin{array}{ccccccc}
 0 \rightarrow & E|_x/E_{x,i} \otimes \mathcal{O}_X(-x)|_x & \rightarrow & E_x^i|_x & \xrightarrow{f} & E|_x & \rightarrow E|_x/E_{x,i} \rightarrow 0 \\
 & \parallel & & \searrow & \nearrow & & \\
 & \text{Tor}(E|_x/E_{x,i}, \mathcal{O}_x) & & & E_{x,i} & & \\
 & & & \nearrow & & \searrow & \\
 & & & 0 & & 0 &
 \end{array}$$

Observe that the tail of the filtration $E_{x,i} \supsetneq E_{x,i+1} \supset E_{x,l_x+1} = 0$ of $E|_x$, induce a filtration of $E_x^i|_x$

$$E_x^i|_x = f^{-1}(E_{x,i}) \supsetneq f^{-1}(E_{x,i+1}) \supsetneq \cdots \supsetneq f^{-1}(E_{x,l_x}) \supsetneq f^{-1}(0) = \frac{E|_x}{E_{x,i}} \otimes \mathcal{O}_X(-x)|_x$$

on the other hand, the head of the filtration $E|_x = E_{x,1} \supsetneq E_{x,2} \supsetneq \cdots \supsetneq E_{x,i}$ induce canonically a filtration on $\frac{E|_x}{E_{x,i}}$

$$\frac{E|_x}{E_{x,i}} = \frac{E_{x,1}}{E_{x,i}} \supsetneq \frac{E_{x,2}}{E_{x,i}} \supsetneq \cdots \supsetneq \frac{E_{x,i}}{E_{x,i}} = 0$$

thus, $E_x^i|_x$ gets an induced filtration at x of the same length as that of $E|_x$

$$E_x^i|_x = f^{-1}(E_{x,i}) \supsetneq \cdots \supsetneq f^{-1}(E_{x,l_x}) \supsetneq \frac{E|_x}{E_{x,i}} \otimes \mathcal{O}_X(-x)|_x \supsetneq \cdots \supsetneq \frac{E_{x,i-1}}{E_{x,i}} \otimes \mathcal{O}_X(-x)|_x \supsetneq 0$$

On the other hand, $E_x^i|_y$ is canonically isomorphic to $E|_y$ for each $y \in D \setminus \{x\}$, thus inheriting its filtration. Therefore, for each $x \in D$ and each $1 \leq i \leq l_x + 1$, we can provide E_x^i a canonical quasi-parabolic structure with the same number of steps as (E, E_\bullet) . In particular, if (E, E_\bullet) is full flag, then the induced quasi-parabolic structure on E_x^i is full flag. This “rotation” procedure – also called by some authors elementary transformation of the parabolic bundle – has been used in the literature as a fruitful way to induce correspondences between moduli spaces of parabolic vector bundles [BY99, IIS06b, Ina13].

We call E_x^2 with the induced parabolic structure the Hecke transformation of (E, E_\bullet) at x , and we will denote it by

$$(E_x^2, (E_x^2)_\bullet) = \mathcal{H}_x(E, E_\bullet)$$

More generally, we will write

$$\mathcal{H}_x^k(E, E_\bullet) = \overbrace{\mathcal{H}_x \circ \cdots \circ \mathcal{H}_x}^k(E, E_\bullet)$$

It is straightforward to check that for each $1 \leq k \leq l_x$ the quasi-parabolic bundle $\mathcal{H}_x^k(E, E_\bullet)$ coincides with the vector bundle E_x^{k+1} with the induced parabolic structure previously described. Also, by construction, for every quasi-parabolic vector bundle and every $x \in D$ the following relation holds

$$\mathcal{H}_x^{l_x}(E, E_\bullet) = (E, E_\bullet) \otimes \mathcal{O}_X(-x)$$

Moreover, it is clear that two Hecke transformations at two different parabolic points commute with each other. Let H denote an effective divisor on X supported on $D = \{x_1, \dots, x_n\}$. If we take $H = \sum_{x \in D} h_x x$, then we define \mathcal{H}_H as the composition

$$\mathcal{H}_H = \mathcal{H}_{x_1}^{h_{x_1}} \circ \mathcal{H}_{x_2}^{h_{x_2}} \circ \dots \circ \mathcal{H}_{x_n}^{h_{x_n}}$$

We can understand the Hecke transformation of a quasi-parabolic vector bundle in another equivalent way working directly over the filtration by subsheaves. Let (E, E_\bullet) be a full flag parabolic vector bundle and let $x \in D$ be a parabolic point. We define the Hecke transformation of (E, E_\bullet) at x to be the parabolic vector bundle $\mathcal{H}_x(E, E_\bullet) = (H, H_\bullet)$ obtained by taking the Hecke transformation of E with respect to $E_{x,2}$ and “rotating” the parabolic structure at x in the following way. We take

$$\begin{aligned} \forall i = 1, \dots, r \quad H_x^i &= E_x^{i+1} \\ \forall y \in D \setminus \{x\} \forall i = 1, \dots, r \quad H_y^i &= \mathcal{H}_x^{E_{x,2}}(E_y^i) \end{aligned}$$

In particular, $H = E_x^2 = \mathcal{H}_x^{E_{x,2}}(E)$. For example, for $r = 3$, $D = x + y$, we send the parabolic vector bundle

$$(E, E_\bullet) = \left\{ \begin{array}{ccccccc} E = E_x^1 & \supsetneq & E_x^2 & \supsetneq & E_x^3 & \supsetneq & E(-x) \\ E = E_y^1 & \supsetneq & E_y^2 & \supsetneq & E_y^3 & \supsetneq & E(-x) \end{array} \right\}$$

to

$$\mathcal{H}_x(E, E_\bullet) = \left\{ \begin{array}{ccccccc} E_x^2 & \supsetneq & E_x^3 & \supsetneq & E(-x) = E_x^1(-x) & \supsetneq & E_x^2(-x) \\ E_x^2 = \mathcal{H}_x^{E_{x,2}}(E_y^1) & \supsetneq & \mathcal{H}_x^{E_{x,2}}(E_y^2) & \supsetneq & \mathcal{H}_x^{E_{x,3}}(E_y^3) & \supsetneq & E_x^2(-x) \end{array} \right\}$$

Observe that if we choose weights α on a full flag quasi parabolic vector bundle (E, E_\bullet) , then we might maintain the same system of weights α on $\mathcal{H}_x(E, E_\bullet)$ and, in that case

$$\text{pardeg}_\alpha(\mathcal{H}_x(E, E_\bullet)) = \text{pardeg}_\alpha(E, E_\bullet) - 1$$

Nevertheless, we will see that this is not a natural choice of weights, as it does not preserve stability.

Lemma 4.4.1. *Suppose that $g \geq 3$. Suppose that $d = \deg(\xi)$ and r are coprime and let α be a generic concentrated system of weights. For every divisor H with $0 < d \leq H \leq (r-1)D$ such that $d < |H|$, there exists at least a stable parabolic vector bundle $(E, E_\bullet) \in \mathcal{M}(r, \alpha, \xi)$ over (X, D) such that $\mathcal{H}_H(E, E_\bullet)$ is α -unstable.*

Proof. Let $d = \deg(\xi)$. By tensoring with an appropriate line bundle, we can assume that $0 \leq d < r$. Brambila-Paz, Grzegorzczuk and Newstead [BPGN97] proved that for every genus $g \geq 2$ smooth projective curve and every $0 \leq d < r$, the space of stable vector bundles E of rank r and degree d such that $H^0(E) \geq k$ (called the Brill-Nether locus and usually denoted by $B(r, d, k)$) is nonempty if $d > 0$ and

$$r \leq d + (r-k)g$$

with $(r, d, k) \neq (r, r, r)$. As we are assuming that d and r are coprime, then $0 < d$ and for $k = 1$ and $g \geq 3$

$$d + (r-k)g - r \geq d + 3(r-1) - r = d + 2r - 3 \geq 2(r-1) > 0$$

Then, there exists a stable vector bundle E with rank r and degree d such that $H^0(E) > 0$. As $H^0(E) > 0$, \mathcal{O}_X is a subsheaf of E and, saturating, there is a line bundle $L \subsetneq E$ with $0 \leq \deg(L) < \mu(E)$. Tensoring with a suitable degree zero line bundle, we might assume that $\det(E) \cong \xi$. The weights α are concentrated and rank and degree are coprime, so the stability of any parabolic vector bundle is equivalent to the stability of its underlying vector bundle. Therefore, for every choice of filtrations over $E|_x$, for $x \in D$, the parabolic vector bundle (E, E_\bullet) is stable. In particular, we can choose a parabolic structure (E, E_\bullet) such that $E_{x,r} = L|_x$ for every $x \in D$.

Then, (E, E_\bullet) is stable and L is a subsheaf of $E_x^k = \mathcal{H}_x^{E_x, k}$ for every $k < r$. Therefore, L is a subsheaf of $\mathcal{H}_H(E, E_\bullet)$. Let \bar{L} be its saturation. Then

$$\deg(\bar{L}) \geq \deg(L) \geq 0$$

On the other hand, as $d < |H|$,

$$\mu(\mathcal{H}_x(E, E_\bullet)) = \frac{d - |H|}{r} \leq \frac{d - d}{r} = 0 \leq \deg(\bar{L})$$

so the underlying vector bundle of $\mathcal{H}_x(E, E_\bullet)$ is unstable. Therefore, as the parabolic weights are concentrated, $\mathcal{H}_x(E, E_\bullet)$ is α -unstable as a parabolic vector bundle. \square

Lemma 4.4.2. *Let X be a smooth complex projective curve of genus g . Let r, s, k, d be integers such that $0 < k < r$. Then if*

$$g > \frac{r - 1 - s}{k} + 1$$

then there exist a stable vector bundle E of degree d and rank r and a subbundle $F \subsetneq E$ of rank k such that

$$kd - r \deg(F) = s$$

Proof. By [BPL98, Remark 3.3], there exists a stable vector bundle E such that $s_k(E) = s$ if for every $1 \leq i < k$

$$0 < i(k - i)(g - 1) - \frac{i}{k}(k(r - k)(g - 1) - s + r - 1)$$

As $k > 0$, multiplying by $k/i > 0$ yields that this is equivalent to proving that

$$0 < k(k - i)(g - 1) - k(r - k)(g - 1) + s - r + 1 = k(k - i)(g - 1) + s - r + 1$$

But, as $1 \leq i < k$ we obtain

$$g > \frac{r - 1 - s}{k} + 1 \geq \frac{r - 1 - s}{k(k - i)} + 1$$

for all $1 \leq i < k$ and the lemma follows. \square

Lemma 4.4.3. *Suppose that $g > 3$. Suppose that $d = \deg(\xi)$ and r are coprime and let α be a generic concentrated system of weights. If H is a divisor with $|H| = 2d - r > 0$, there exists at least a stable parabolic vector bundle $(E, E_\bullet) \in \mathcal{M}(r, \alpha, \xi)$ over (X, D) such that $\mathcal{H}_H(E, E_\bullet)$ is α -unstable.*

Proof. As $0 < |H| = 2d - r$, we have $d > r/2$. In particular, as we assumed $r > d$, then $r \geq 3$. For every vector bundle E

$$\mu(\mathcal{H}_H(E)) = \frac{d - |H|}{r} = \frac{d - (2d - r)}{r} = \frac{r - d}{r} = 1 - \mu(E)$$

Let $k = r - |H| = r - (2d - r) = 2(r - d)$ and let

$$d' = \left\lceil \frac{2(r - d)^2}{r} \right\rceil$$

Then by Lemma 4.4.4 below

$$\frac{r - d}{r} < \frac{d'}{k} \leq \frac{d}{r} \quad (4.4.2)$$

Assume that there exists a stable vector bundle E of rank r and degree d with a subbundle $F \subseteq E$ of rank $k = r - |H| = r - (2d - r) = 2(r - d)$ and degree d' . As the parabolic weights are concentrated and rank and degree are coprime, parabolic stability is equivalent to stability of the underlying bundle for any choice of the filtrations E_\bullet . We have $\text{rk}(F) = r - |H|$, so we can choose a parabolic structure (E, E_\bullet) on E such that $F|_x = E_{x, |H|}$ for every $x \in D$. Therefore, F is a subsheaf of $\mathcal{H}_H(E, E_\bullet)$. By inequality (4.4.2), the saturation of F in $\mathcal{H}_H(E, E_\bullet)$ is a destabilizing subsheaf of the underlying vector bundle of $\mathcal{H}_H(E, E_\bullet)$. As the weights are concentrated, then $\mathcal{H}_H(E, E_\bullet)$ is α -unstable as a parabolic vector bundle.

In order to find the desired E and F we can apply Lemma 4.4.2 for $k = 2(r - d)$ and $s = kd - rd'$. To guarantee the genus hypothesis of the Lemma, it is enough to show that

$$g > 3 \geq \frac{r - 1 - s}{2(r - d)} + 1$$

Using the bound $\lceil x \rceil < x + 1$ on the d' formula yields

$$\begin{aligned} \frac{r - 1 - s}{2(r - d)} + 1 &= \frac{r - 1 - 2(r - d)d + rd'}{2(r - d)} + 1 = \frac{r - 1}{2(r - d)} - d + 1 + \frac{r \left\lceil \frac{2(r - d)^2}{r} \right\rceil}{2(r - d)} \\ &< \frac{r - 1}{2(r - d)} - d + 1 + \frac{2(r - d)^2 + r}{2(r - d)} = \frac{2r - 1}{2(r - d)} + r - 2d + 1 < \frac{r}{r - d} + r - 2d + 1 \end{aligned}$$

Multiplying by $r - d$ and reordering the factors, the desired inequality is then equivalent to

$$r + (r - d)(r - 2d) - 2(r - d) = r - (r - d)(2d - r + 2) \leq 0$$

Let

$$\begin{cases} r = 2\bar{r} \\ d = \bar{r} + \varepsilon \end{cases}$$

Substituting in the above expression and reordering yields

$$\begin{aligned} r - (r - d)(2d - r + 2) &= 2\bar{r} - (2\bar{r} - \bar{r} - \varepsilon)(2(\bar{r} + \varepsilon) - 2\bar{r} + 2) \\ &= 2\bar{r} - 2(\bar{r} - \varepsilon)(\varepsilon + 1) = 2\varepsilon(\bar{r} - \varepsilon - 1) \end{aligned}$$

which is clearly nonnegative, as $\varepsilon > 0$ and $\bar{r} - \varepsilon = r - d \geq 1$. □

Lemma 4.4.4. *If $r \geq 3$ and $0 < r - d < d < r$ then*

$$\frac{2d(r-d)}{r} \geq \left\lceil \frac{2(r-d)^2}{r} \right\rceil$$

Proof. Let $\varepsilon = d - r/2$. Then clearly $0 < \varepsilon < r/2$ and we have

$$\begin{cases} r - d = \frac{r}{2} - \varepsilon \\ d = \frac{r}{2} + \varepsilon \end{cases}$$

It is enough to prove that $\frac{2d(r-d)}{r} \geq \frac{2(r-d)^2}{r} + 1$. Then we might rewrite the inequality as

$$\frac{r}{2} - 2\frac{\varepsilon^2}{r} = \frac{2d(r-d)}{r} \geq \frac{2(r-d)^2}{r} + 1 = \frac{r}{2} - 2\varepsilon + 2\frac{\varepsilon^2}{r} + 1$$

which is equivalent to

$$\frac{4}{r}\varepsilon^2 - 2\varepsilon + 1 \leq 0$$

The roots of the above quadratic polynomial in ε are

$$\varepsilon_{\pm} = \frac{r}{4} \pm \frac{1}{2}\sqrt{r^2 - 4}$$

As $\frac{4}{r} > 0$, in order to prove the Lemma it is enough to show that $\varepsilon_- \leq 0 < r/2 \leq \varepsilon_+$, which is equivalent to

$$\frac{r}{4} \leq \frac{1}{2}\sqrt{r^2 - 4}$$

Solving for $r > 0$ yields $r > 4/\sqrt{3}$, and this is satisfied for every $r \geq 3$. \square

Notice that preserving the same system of weights on the Hecke transformation is not a natural choice, but rather an imposition if we want to restrict ourselves to analyzing the stability with respect to a fixed set of parameters α . In fact, if the “rotation” operation on the parabolic structure is held at the continuous filtration level, the following parabolic weights arise as the natural ones on $\mathcal{H}_x(E, E_{\bullet})$.

Given a system of weights α over (X, D) and a divisor $H = \sum_{x \in D} h_x x$ with $0 \leq H \leq (r-1)D$ we define $\mathcal{H}_H(\alpha)$ to be the set of parameters satisfying

$$\mathcal{H}_H(\alpha)_i(x) = \begin{cases} \alpha_{i+h_x}(x) - \alpha_{1+h_x}(x) & i + h_x \leq r \\ \alpha_{i+h_x-r}(x) - \alpha_{1+h_x}(x) + 1 & i + h_x > r \end{cases}$$

Let us prove that if a parabolic vector bundle (E, E_{\bullet}) is α -stable, then its Hecke transformation $\mathcal{H}_H(E, E_{\bullet})$ is $\mathcal{H}_H(\alpha)$ -stable. In order to do so, we will give yet another interpretation of the Hecke transformation in terms of the parabolic tensor product.

Let $0 < H \leq (r-1)D$ be an effective divisor, and let $\varepsilon(x) \in [0, 1)$ be real numbers indexed by $x \in D$ such that

$$\alpha_{h_x}(x) < \varepsilon(x) \leq \alpha_{1+h_x}(x)$$

Let $(\mathcal{O}_X(-D), \mathcal{O}_{X, \bullet}(-D)^{1-\varepsilon})$ be the parabolic line bundle obtained by giving $\mathcal{O}_X(-D)$ the trivial filtration with weight $1 - \varepsilon(x)$ at $x \in D$. Consider the parabolic vector

bundle $(H, H_\bullet) = (E, E_\bullet) \otimes (\mathcal{O}_X(-D), \mathcal{O}_{X,\bullet}(-D)^{1-\varepsilon})$. By construction, for every $x \in D$ and every $\alpha \in \mathbb{R}$

$$H_\alpha^x = E_{\alpha+\varepsilon(x)}^x$$

In particular, for $\alpha = 0$, one gets

$$H_0^x = E_{\varepsilon(x)}^x = E_{\alpha_1+h_x(x)}^x = E_x^{1+h_x}$$

Therefore H is the underlying vector bundle of $\mathcal{H}_H(E, E_\bullet)$. A similar computation shows that, in fact

$$\mathcal{H}_H(E, E_\bullet) = (E, E_\bullet) \otimes (\mathcal{O}_X(-D), \mathcal{O}_{X,\bullet}(-D)^{1-\varepsilon})$$

as quasi-parabolic vector bundles. Let us prove that for each admissible choice of ε , the right hand side is a stable parabolic vector bundle.

Proposition 4.4.5. *Let (E, E_\bullet) be a rank r full flag parabolic vector bundle with system of weights α , and let $(L, L_\bullet^\varepsilon)$ be any parabolic line bundle with system of weights ε . Then*

$$\text{pardeg}((E, E_\bullet) \otimes (L, L_\bullet^\varepsilon)) = \text{pardeg}(E, E_\bullet) + r \text{pardeg}(L, L_\bullet^\varepsilon)$$

Proof. For each $x \in D$ Let h_x be the only integer such that

$$\alpha_{h_x}(x) < 1 - \varepsilon(x) \leq \alpha_{1+h_x}(x)$$

Let $(H, H_\bullet) = (E, E_\bullet) \otimes (L, L_\bullet^\varepsilon)$, and let β be the induced system of weights for (H, H_\bullet) . Then, by construction,

$$\beta_i(x) = \begin{cases} \alpha_{i+h_x}(x) + \varepsilon(x) - 1 & i + h_x \leq r \\ \alpha_{i+h_x-r}(x) + \varepsilon(x) & i + h_x > r \end{cases}$$

The degree of H is

$$\deg(H) = \deg(E) + r \deg(L) + \sum_{x \in D} (r - h_x)$$

On the other hand,

$$\begin{aligned} \text{wt}(H, H_\bullet) &= \sum_{x \in D} \sum_{i=1}^{l_x} \beta_i(x) = \sum_{x \in D} \sum_{i=1}^{r-h_x} (\alpha_{i+h_x}(x) + \varepsilon(x) - 1) \\ &\quad + \sum_{x \in D} \sum_{i=r-h_x+1}^r (\alpha_{i+h_x-r}(x) + \varepsilon(x)) = \text{wt}_\alpha(E, E_\bullet) + r \sum_{x \in D} \varepsilon(x) - \sum_{x \in D} (r - h_x) \end{aligned}$$

Adding up

$$\begin{aligned} \text{pardeg}(H, H_\bullet) &= \deg(H) + \text{wt}(H, H_\bullet) = \deg(E) + r \deg(L) + \text{wt}(E, E_\bullet) + r \sum_{x \in D} \varepsilon(x) \\ &= \text{pardeg}(E, E_\bullet) + r \text{pardeg}(L) \end{aligned}$$

□

Corollary 4.4.6. *Let (E, E_\bullet) be a (semi)-stable parabolic vector bundle with system of weights α , and let $(L, L_\bullet^\varepsilon)$ be a parabolic line bundle with system of weights ε . Then $(E, E_\bullet) \otimes (L, L_\bullet^\varepsilon)$ is stable for the induced parabolic structure.*

Proof. We have that $(F, F_\bullet) \subset (E, E_\bullet)$ if and only if $(F, F_\bullet) \otimes (L, L_\bullet^\varepsilon) \subset (E, E_\bullet) \otimes (L, L_\bullet^\varepsilon)$, and, by the previous Proposition

$$\text{pardeg}((F, F_\bullet) \otimes (L, L_\bullet^\varepsilon)) = \text{pardeg}(F, F_\bullet) + \text{rk}(F) \text{pardeg}(L, L_\bullet^\varepsilon)$$

$$\text{pardeg}((E, E_\bullet) \otimes (L, L_\bullet^\varepsilon)) = \text{pardeg}(E, E_\bullet) + \text{rk}(E) \text{pardeg}(L, L_\bullet^\varepsilon)$$

Therefore,

$$\frac{\text{pardeg}((E, E_\bullet) \otimes (L, L_\bullet^\varepsilon))}{\text{rk}(E)} - \frac{\text{pardeg}((F, F_\bullet) \otimes (L, L_\bullet^\varepsilon))}{\text{rk}(F)} = \frac{\text{pardeg}(E, E_\bullet)}{\text{rk}(E)} - \frac{\text{pardeg}(F, F_\bullet)}{\text{rk}(F)}$$

so $(E, E_\bullet) \otimes (L, L_\bullet^\varepsilon)$ is (semi)stable if and only if (E, E_\bullet) is (semi)stable. \square

Corollary 4.4.7. *A full flag parabolic vector bundle (E, E_\bullet) is α -(semi)stable if and only if $\mathcal{H}_H(E, E_\bullet)$ is $\mathcal{H}_H(\alpha)$ -(semi)stable.*

Thus, Hecke transformations preserve stability with respect to the natural induced system of weights, but Lemmas 4.4.1 and 4.4.3 show that the induced system $\mathcal{H}_H(\alpha)$ might not belong to the same stability chamber as the original one α .

We can also describe an analogue of dualization in the quasi-parabolic context. Given a quasi-parabolic vector bundle (E, E_\bullet) described as a set of decreasing filtrations

$$E|_x = E_{x,1} \supsetneq E_{x,2} \supsetneq \cdots \supsetneq E_{x,l_r} \supsetneq 0$$

for each $x \in D$, observe that if we take the dual of the corresponding spaces then we obtain

$$E^\vee|_x = E_{x,1}^\vee \twoheadrightarrow E_{x,2}^\vee(-x) \twoheadrightarrow \cdots \twoheadrightarrow (E_{x,l_r})^\vee(-x) \twoheadrightarrow 0$$

taking the kernels of the successive quotients (i.e., taking the corresponding annihilators in $E^\vee|_x$) we obtain

$$E^\vee|_x = \text{ann}(0) \supsetneq \text{ann}(E_{x,l_r}) \supsetneq \cdots \supsetneq \text{ann}(E_{x,2}) \supsetneq \text{ann}(E_{x,1}) = 0$$

which clearly provides us a quasi-parabolic structure over E^\vee with the same number of steps. We will denote the vector bundle E^\vee with this induced quasi-parabolic structure as $(E, E_\bullet)^\vee$ and we will call it its quasi-parabolic dual. Observe that if (E, E_\bullet) is full flag, then $(E, E_\bullet)^\vee$ is also full flag. Notice that this definition of dual is different to the usual notion of parabolic dual of a parabolic vector bundle, described, for example in [Bis03] (see Section B.3.4 in Appendix B for full details). Let us fix a system of weights α for (E, E_\bullet) . Biswas defines the parabolic dual of the parabolic vector bundle (E, E_\bullet) in terms of the left continuous decreasing filtrations E_α as the parabolic vector bundle $(E, E_\bullet)^*$ obtained by

$$((E, E_\bullet)^*)_\alpha = \lim_{t \rightarrow \alpha^+} (E_{-1-t})^*$$

It is clear that the underlying vector bundle of $(E, E_\bullet)^*$ does not, in general, coincide with E^\vee . In fact, the underlying vector bundle $((E, E_\bullet)^*)_0$ depends on the choice

of the parabolic weights α . More precisely, they depend on whether $\alpha_1(x) = 0$ for the points $x \in D$. If $\alpha_1(x) > 0$ for each $x \in D$, we have

$$((E, E_\bullet)^*)_0 = (E_{-1})^\vee = E^\vee(-D)$$

In this case, it can be checked that the induced filtration on $E^\vee(-D)$ is precisely the one obtained by tensoring $(E, E_\bullet)^\vee$ with $\mathcal{O}_X(-D)$. One of the main advantages of the latter approach in conjunction to the definition of parabolic tensor product is that it allows us to work with sheaves of parabolic morphism in a way similar to the one used for regular vector bundles, as the sheaf of parabolic morphism (morphisms preserving the parabolic structure) from (E, E_\bullet) to (F, F_\bullet) simply becomes

$$\mathrm{PHom}((E, E_\bullet), (F, F_\bullet)) = (E, E_\bullet)^* \otimes (F, F_\bullet)$$

Suppose that α is a full flag system of weights with $\alpha_1(x) > 0$ for all $x \in D$. If (E, E_\bullet) is a stable (respectively semi-stable) parabolic vector bundle, then $(E, E_\bullet)^*$ is stable (respectively semi-stable) with respect to the following system of weights α^\vee

$$\alpha_i^\vee(x) = 1 - \alpha_i(x)$$

Under these hypothesis on α , $(E, E_\bullet)^* = (E, E_\bullet)^\vee \otimes \mathcal{O}_X(-D)$ as quasi-parabolic vector bundles, so we just saw that if $\alpha_1(x) > 0$ for all $x \in D$, then (E, E_\bullet) is α -stable if and only if $(E, E_\bullet)^\vee$ is α^\vee -stable.

Notice that, in particular, if the system of weights α is concentrated, then α^\vee is also concentrated, so in the concentrated chamber α -stability is equivalent to α^\vee -stability.

Up to this point, we have studied three types of operations that can be performed on quasi-parabolic vector bundles (E, E_\bullet) and the corresponding transformations on the systems of weights that must be done to ensure stability of the resulting parabolic vector bundle

- Tensor with a line bundle $(E, E_\bullet) \mapsto (E, E_\bullet) \otimes L$
- Dualization $(E, E_\bullet) \mapsto (E, E_\bullet)^\vee$
- Hecke transformations $(E, E_\bullet) \mapsto \mathcal{H}_H(E, E_\bullet)$

Moreover, if (E, E_\bullet) is a parabolic α -(semi)stable vector bundle and $\sigma : X \rightarrow X$ is an automorphism of X that sends D to itself (not necessarily fixing each parabolic point), then the pullback $\sigma^*(E, E_\bullet)$ is a $\sigma^*\alpha$ -(semi)stable parabolic vector bundle, where

$$\sigma^*\alpha_i(x) = \alpha_i(\sigma^{-1}(x))$$

These four transformations can be clearly extended canonically to families of α -(semi)stable parabolic vector bundles, so we will denote the combinations of them as “basic” transformations of quasi-parabolic vector bundles.

Definition 4.4.8. *Let (X, D) be a Riemann surface with a set of marked points $D \subset X$. A basic transformation of a quasi-parabolic vector bundle is a tuple $T = (\sigma, s, L, H)$ consisting on*

- An automorphism $\sigma : X \xrightarrow{\sim} X$ that sends D to itself (but does not necessarily fix any point of D)
- A sign $s \in \{1, -1\}$.
- A line bundle L on X .
- A divisor H on X such that $0 \leq H \leq (r-1)D$.

Given a quasi-parabolic vector bundle (E, E_\bullet) and a basic transformation $T = (\sigma, s, L, H)$, let

$$T(E, E_\bullet) = \begin{cases} \sigma^*(L \otimes \mathcal{H}_H(E, E_\bullet)) & s = 1 \\ \sigma^*(L \otimes \mathcal{H}_H(E, E_\bullet))^\vee & s = -1 \end{cases}$$

If ξ is a line bundle, we define

$$T(\xi) = \begin{cases} \sigma^*(L^r \otimes \xi(-H)) & s = 1 \\ \sigma^*(L^r \otimes \xi(-H))^\vee & s = -1 \end{cases}$$

Finally, if α is a rank r system of weights over (X, D) , we define

$$T(\alpha)_i(x) = \begin{cases} \mathcal{H}_H(\alpha)_i(\sigma^{-1}(x)) & s = 1 \\ 1 - \mathcal{H}_H(\alpha)_{r-i+1}(\sigma^{-1}(x)) & s = -1 \end{cases}$$

Observe that the action of T on the space of admissible systems of weights is stable under translations of the system in the following sense. Let $\varepsilon = (\varepsilon(x))_{x \in D} \in \mathbb{R}^{|D|}$ such that for every $x \in D$ $-\alpha_1(x) \leq \varepsilon(x) < 1 - \alpha_r(x)$. Consider the system of weights $\alpha[\varepsilon]$ defined as

$$\alpha[\varepsilon]_i(x) = \alpha_i(x) + \varepsilon(x)$$

we call $\alpha[\varepsilon]$ the translation of α by ε .

Then for any admissible type of a subbundle \bar{n}'

$$\begin{aligned} s_{\min}(\alpha[\varepsilon], \bar{n}) &= r'' \sum_{x \in D} \sum_{i=1}^r n'_i(x)(\alpha_i(x) + \varepsilon(x)) - r' \sum_{i=1}^r n''_i(x)(\alpha_i(x) + \varepsilon(x)) \\ &= r'' \sum_{x \in D} \sum_{i=1}^r n'_i(x)\alpha_i(x) - r' \sum_{i=1}^r n''_i(x)\alpha_i(x) = s_{\min}(\alpha, \bar{n}) \end{aligned}$$

Therefore, α -stability is completely equivalent to $\alpha[\varepsilon]$ -stability. Let

$$\Delta = \{\alpha = (\alpha_i(x)) \in [0, 1)^{r|D|} \mid \forall x \in D \forall i = 1, \dots, r-1 \alpha_i(x) < \alpha_{i+1}(x)\}$$

be the space of systems of weights over (X, D) , and let $\Delta^+ = \Delta \cap (0, 1)^{r|D|}$. Let us define an equivalence relation \sim on Δ as follows. $\alpha \sim \beta$ if and only if there exists some $\varepsilon = (\varepsilon(x))_{x \in D}$ such that for every $x \in D$ we have

$$-\alpha_1(x) \leq \varepsilon(x) < 1 - \alpha_r(x)$$

and such that $\beta = \alpha[\varepsilon]$. Define $\tilde{\Delta}$ as the quotient Δ / \sim . Clearly $\Delta / \sim = \Delta^+ / \sim$.

Let $\alpha, \beta \in \Delta^0$ such that $\alpha \sim \beta$. Then for every basic transformation T we have $T(\alpha) \sim T(\beta)$. Therefore, basic transformations act on $\tilde{\Delta}$. In particular, in $\tilde{\Delta}$ for every $x \in D$ and every $\alpha \in \Delta$

$$\mathcal{H}_x^r(\alpha) \sim \alpha$$

By construction (E, E_\bullet) is an α -(semi)stable parabolic vector bundle with determinant ξ if and only if $T(E, E_\bullet)$ is an $T(\alpha)$ -(semi)stable parabolic vector bundle with determinant $T(\xi)$.

Basic transformations form a group \mathcal{T} , where the product rule is the composition. We can give an explicit natural presentation, which is independent on whether we are making \mathcal{T} act on quasi-parabolic vector bundles, line bundles or weight systems. It is generated by

- $\Sigma_\sigma = (\sigma, 1, \mathcal{O}_X, 0)$
- $\mathcal{D}^+ = (\text{Id}, 1, \mathcal{O}_X, 0) = \text{Id}_{\mathcal{T}}$
- $\mathcal{D}^- = (\text{Id}, -1, \mathcal{O}_X, 0)$
- $\mathcal{T}_L = (\text{Id}, 1, L, 0)$
- $\mathcal{H}_H = (\text{Id}, 1, \mathcal{O}_X, H)$

And we have the following composition rules

1. $\Sigma_\sigma \circ \Sigma_\tau = \Sigma_{\sigma \circ \tau}$
2. $\mathcal{D}^s \circ \mathcal{D}^t = \mathcal{D}^{st}$
3. $\mathcal{T}_L \circ \mathcal{T}_M = \mathcal{T}_{L \otimes M}$
4. If $0 \leq H_i \leq (r-1)D$ for $i = 1, 2$ then

$$\mathcal{H}_{H_1} \circ \mathcal{H}_{H_2} = \mathcal{T}_{L_{H_1+H_2}} \circ \mathcal{H}_{H_1+H_2-L_{H_1+H_2}}$$

where, given a divisor $F = \sum_{x \in D} f_x x$, we define

$$L_F = \sum_{x \in D} \left\lfloor \frac{f_x}{r} \right\rfloor x$$

5. $\Sigma_\sigma \circ \mathcal{D}^s = \mathcal{D}^s \circ \Sigma_\sigma$
6. $\Sigma_\sigma \circ \mathcal{T}_L = \mathcal{T}_{\sigma^* L} \circ \Sigma_\sigma$
7. $\Sigma_\sigma \circ \mathcal{H}_H = \mathcal{H}_{\sigma^* H} \circ \Sigma_\sigma$
8. $\mathcal{D}^- \circ \mathcal{T}_L = \mathcal{T}_{L^{-1}} \circ \mathcal{D}^-$
9. $\mathcal{D}^- \circ \mathcal{H}_H = \mathcal{T}_{\mathcal{O}_X(D)} \circ \mathcal{H}_{rD-H} \circ \mathcal{D}^-$, for $H > 0$
10. $\mathcal{T}_L \circ \mathcal{H}_H = \mathcal{H}_H \circ \mathcal{T}_L$

From these composition rules, it is straightforward to compute the inverses of each generator

- $\Sigma_\sigma^{-1} = \Sigma_{\sigma^{-1}}$
- $(\mathcal{D}^s)^{-1} = \mathcal{D}^s$
- $\mathcal{T}_L^{-1} = \mathcal{T}_{L^{-1}}$
- $\mathcal{H}_H^{-1} = \mathcal{T}_{\mathcal{O}_X(D)} \circ \mathcal{H}_{rD-H}$ for $H > 0$.

Then, using the composition rules it is easy to check that the inverse of a basic transformation $T = (\sigma, s, L, H)$ for $H > 0$ is

$$\begin{aligned}
T^{-1} &= \mathcal{H}_H^{-1} \circ \mathcal{T}_L^{-1} \circ (\mathcal{D}^s)^{-1} \circ \Sigma_\sigma^{-1} = \mathcal{T}_{L^{-1}(D)} \circ \mathcal{H}_{rD-H} \circ \mathcal{D}^s \circ \Sigma_{\sigma^{-1}} \\
&= \left\{ \begin{array}{ll} \mathcal{D}^+ \circ T_{L^{-1}(D)} \circ \mathcal{H}_{rD-H} \circ \Sigma_{\sigma^{-1}} & s = 1 \\ \mathcal{D}^- \circ T_L \circ \mathcal{H}_H \circ \Sigma_{\sigma^{-1}} & s = -1 \end{array} \right\} \\
&= \left\{ \begin{array}{ll} \Sigma_{\sigma^{-1}} \circ \mathcal{D}^+ \circ ST_{\sigma^*L^{-1}(D)} \circ \mathcal{H}_{rD-\sigma^*H} & s = 1 \\ \Sigma_{\sigma^{-1}} \circ \mathcal{D}^- \circ \mathcal{T}_{\sigma^*L} \circ \mathcal{H}_{\sigma^*H} & s = -1 \end{array} \right\} \\
&= \left\{ \begin{array}{ll} (\sigma^{-1}, 1, \sigma^*L^{-1}(D), rD - \sigma^*H) & s = 1 \\ (\sigma^{-1}, -1, \sigma^*L, \sigma^*H) & s = -1 \end{array} \right\}
\end{aligned}$$

And the inverse for $H = 0$ is

$$(\sigma, s, L, 0)^{-1} = \left\{ \begin{array}{ll} (\sigma^{-1}, 1, \sigma^*L^{-1}, 0) & s = 1 \\ (\sigma^{-1}, -1, \sigma^*L, 0) & s = -1 \end{array} \right\} = (\sigma^{-1}, s, \sigma^*L^{-s}, 0)$$

With this presentation we can describe the abstract group structure of \mathcal{T} .

Proposition 4.4.9. *The group of basic transformations is isomorphic to the following semidirect product*

$$\mathcal{T} \cong \left((\mathbb{Z}^{|D|} \times \text{Pic}(X)) / \mathcal{G}_D \right) \rtimes (\text{Aut}(X, D) \times \mathbb{Z}/2\mathbb{Z})$$

where

$$\mathcal{G}_D = \{(rH, \mathcal{O}_X(H)) \mid H \text{ supported on } D\} < \mathbb{Z}^{|D|} \times \text{Pic}(X)$$

Proof. Let us consider the surjective map $\pi : \mathcal{T} \rightarrow \langle \mathcal{D}^-, \mathcal{T}_L \rangle$ which sends a basic transformation (σ, s, L, H) to $\Sigma_\sigma \circ \mathcal{D}^s$. Let us prove that it is a group homomorphism. Let (σ, s, L, H) and (σ', s', L', H') be basic transformations. Then

$$(\sigma, s, L, H) \circ (\sigma', s', L', H') = (\sigma, s, \mathcal{O}_X, 0) \circ \mathcal{T}_L \circ \mathcal{T}_H \circ \Sigma_{\sigma'} \circ \mathcal{D}^{s'} \circ (\text{Id}, 1, L', H')$$

On the other hand, by properties (6) and (7), there exists L_1 and H_1 such that

$$\mathcal{T}_L \circ \mathcal{T}_H \circ \Sigma_{\sigma'} = \Sigma_{\sigma'} \circ \mathcal{T}_{L_1} \circ \mathcal{T}_{H_1}$$

Similarly, by properties (8), (9) and (10) there exists L_2 and H_2 such that

$$\mathcal{T}_{L_1} \circ \mathcal{T}_{H_1} \circ \mathcal{D}^{s'} = \mathcal{D}^{s'} \circ \mathcal{T}_{L_2} \circ \mathcal{H}_{H_2}$$

So we obtain that

$$\begin{aligned}
(\sigma, s, L, H) \circ (\sigma', s', L', H') &= (\sigma, s, \mathcal{O}_X, 0) \circ (\text{Id}, 1, L, H) \circ (\sigma', s, \mathcal{O}_X, 0) \circ (\text{Id}, 1, L', H') \\
&= (\sigma, s, \mathcal{O}_X, 0) \circ (\sigma', s', \mathcal{O}_X, 0) \circ (\text{Id}, 1, L_2, H_2) \circ (\text{Id}, 1, L', H')
\end{aligned}$$

Finally, applying (1)-(5) and property (10) we have that there exist L_3 and H_3 such that

$$(\sigma, s, \mathcal{O}_X, 0) \circ (\sigma', s', \mathcal{O}_X, 0) \circ (\text{Id}, 1, L_2, H_2) \circ (\text{Id}, 1, L', H') = (\sigma\sigma', ss', L_3, H_3)$$

Therefore

$$\pi((\sigma, s, L, H) \circ (\sigma', s', L', H')) = (\sigma\sigma', ss', \mathcal{O}_X, 0) = \pi(\sigma, s, L, H) \circ \pi(\sigma', s', L', H')$$

The kernel of this map coincides clearly with the subgroup $\langle \mathcal{T}_L, \mathcal{H}_H \rangle < \mathcal{T}$ generated by \mathcal{T}_L and \mathcal{H}_H , so it is normal and we have that

$$\mathcal{T} \cong \langle \mathcal{T}_L, \mathcal{T}_H \rangle \rtimes \langle \Sigma_\sigma, \mathcal{D}^- \rangle$$

On the other hand, by property (5) we know that Σ_σ and \mathcal{D}^- commute, so

$$\langle \Sigma_\sigma, \mathcal{D}^- \rangle \cong \text{Aut}(X, x) \times \mathbb{Z}/2\mathbb{Z}$$

Therefore, we conclude that

$$\mathcal{T} \cong \langle \mathcal{T}_L, \mathcal{T}_H \rangle \rtimes (\text{Aut}(X, x) \times \mathbb{Z}/2\mathbb{Z}) \quad (4.4.3)$$

Finally, let us consider the following group

$$\mathcal{G}_D = \{(rH, \mathcal{O}_X(H)) \mid H \text{ supported on } D\} < \mathbb{Z}^{|D|} \times \text{Pic}(X)$$

As generators \mathcal{H}_x for $x \in D$ and \mathcal{T}_L commute and $\mathcal{H}_x^r = \mathcal{T}_{\mathcal{O}_X(-x)}$ then we have

$$\langle \mathcal{T}_L, \mathcal{H}_H \rangle \cong (\mathbb{Z}^{|D|} \times \text{Pic}(X)) / \mathcal{G}_D$$

Combining this with equation (4.4.3) the Proposition follows. \square

Finally, we briefly describe the analogues of these constructions for projective parabolic bundles. Given a parabolic projective bundle $(\mathbb{P}, \mathbb{P}_\bullet)$, let (E, E_\bullet) be a reduction (it always exists by Lemma 4.1.9)

$$(\mathbb{P}, \mathbb{P}_\bullet) \cong (\mathbb{P}(E), \mathbb{P}(E_\bullet))$$

we define

$$\begin{aligned} (\mathbb{P}, \mathbb{P}_\bullet)^\vee &= \mathbb{P}((E, E_\bullet)^\vee) \\ \mathcal{H}_H(\mathbb{P}, \mathbb{P}_\bullet) &= \mathbb{P}(\mathcal{H}_H(E, E_\bullet)) \end{aligned}$$

Any two reductions are related by tensorization with a line bundle. If L is a line bundle, then

$$\begin{aligned} ((E, E_\bullet) \otimes L)^\vee &= (E, E_\bullet)^\vee \otimes L^\vee \\ \mathcal{H}_H((E, E_\bullet) \otimes L) &= \mathcal{H}_H(E, E_\bullet) \otimes L \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{P}(((E, E_\bullet) \otimes L)^\vee) &= \mathbb{P}((E, E_\bullet)^\vee) \\ \mathbb{P}(\mathcal{H}_H((E, E_\bullet) \otimes L)) &= \mathbb{P}(\mathcal{H}_H(E, E_\bullet)) \end{aligned}$$

So the definition of the dual or Hecke transformations are independent of the choice of the reduction.

4.5 The algebra of parabolic endomorphisms

Let P be the parabolic subgroup of $\mathrm{GL}(r, \mathbb{C})$ consisting on upper triangular matrices. Let \mathcal{S} and \mathcal{G} be the group schemes over X given by the following short exact sequences.

$$0 \rightarrow \mathcal{S} \rightarrow \mathrm{SL}(r, \mathbb{C}) \times X \rightarrow (\mathrm{SL}(r, \mathbb{C})/P) \otimes \mathcal{O}_D \rightarrow 0$$

$$0 \rightarrow \mathcal{G} \rightarrow \mathrm{GL}(r, \mathbb{C}) \times X \rightarrow (\mathrm{GL}(r, \mathbb{C})/P) \otimes \mathcal{O}_D \rightarrow 0$$

Let $\mathfrak{par}\mathfrak{sl} = \mathrm{Lie}(\mathcal{S})$ and $\mathfrak{par}\mathfrak{gl} = \mathrm{Lie}(\mathcal{G})$ denote the sheaves of Lie algebras of \mathcal{S} and \mathcal{G} respectively. Let $\mathrm{Aut}(\mathfrak{par}\mathfrak{sl})$ be the sheaf of groups of local algebra automorphisms of $\mathfrak{par}\mathfrak{sl}$. Let $\mathrm{Inn}(\mathfrak{par}\mathfrak{sl})$ be the subsheaf of inner automorphisms, i.e., the image of the adjoint action $\mathrm{Ad} : \mathcal{S} \rightarrow \mathrm{Aut}(\mathfrak{par}\mathfrak{sl})$. Let $\mathrm{GL}(\mathfrak{par}\mathfrak{sl})$ be the sheaf of local linear automorphisms of $\mathfrak{par}\mathfrak{sl}$ as a vector bundle. Analogous notations will be used for $\mathfrak{par}\mathfrak{gl}$.

As \mathcal{S} is a group scheme over X , $\mathrm{Aut}(\mathfrak{par}\mathfrak{sl})$ is a group scheme over X and $\mathrm{Inn}(\mathfrak{par}\mathfrak{sl})$ is a sub-group scheme over X .

Before engaging the main classification Lemma (Lemma 4.5.14), let us prove some necessary results about linear maps of algebras of matrices. Through this section, given a ring R , let $\mathrm{Mat}_{n \times m}(R)$ be the R -module of $n \times m$ matrices with entries in R .

Lemma 4.5.1. *Let R be a commutative unique factorization domain (UFD). Let $M = (m_{ij}) \in \mathrm{Mat}_{n \times m}(R)$ be a matrix with entries in R . Then all the 2×2 minors of M have null determinant in R if and only if there exist matrices $A = (a_i) \in \mathrm{Mat}_{n \times 1}(R)$ and $B = (b_i) \in \mathrm{Mat}_{1 \times m}(R)$ such that $M = AB$.*

Proof. If $M = AB$, then for every pair (i, j) , $m_{ij} = a_i b_j$. Therefore, for every $i, k \in [1, n]$ and $j, l \in [1, m]$ with $i < k$ and $j < l$

$$\begin{vmatrix} m_{ij} & m_{il} \\ m_{kj} & m_{kl} \end{vmatrix} = \begin{vmatrix} a_i b_j & a_i b_l \\ a_k b_j & a_k b_l \end{vmatrix} = a_i b_j a_k b_l - a_i b_l a_k b_j = 0$$

On the other hand, suppose that every 2×2 minor in M has zero determinant. If M is the zero matrix, it is the product of two zero vectors. Otherwise, let m_{ij} be a nonzero element of M . By reordering rows and columns (i.e., permuting the elements of A and B), we can assume without loss of generality that $m_{11} \neq 0$. Then for every $i, j > 1$

$$\begin{vmatrix} m_{11} & m_{1j} \\ m_{i1} & m_{ij} \end{vmatrix} = 0$$

Therefore $m_{11}m_{ij} = m_{i1}m_{1j}$. R is a GCD domain, so great common divisors exist and are unique up to product by units. Then $m_{11} | \mathrm{GCD}_{j>1}(m_{i1}m_{1j}) = m_{i1} \mathrm{GCD}_{j>1}(m_{1j})$ for every $i > 1$. We conclude that

$$m_{11} | \mathrm{GCD}_{i>1}(m_{i1} \mathrm{GCD}_{j>1}(m_{1j})) = \mathrm{GCD}_{i>1}(m_{i1}) \mathrm{GCD}_{j>1}(m_{1j})$$

As R is a UFD, there exists a decomposition $m_{11} = a_1 b_1$ such that $a_1 | \mathrm{GCD}_{j>1}(m_{1j})$ and $b_1 | \mathrm{GCD}_{i>1}(m_{i1})$. As $a_1 | m_{1j}$ for every $j > 1$, there must exist an element $b_j \in R$ such that $m_{1j} = a_1 b_j$. Similarly, for every $i > 1$, $b_1 | m_{i1}$, so there must

exist an element $a_i \in R$ such that $m_{i1} = a_i b_1$. Finally, for every $i, j > 1$, as $m_{11}m_{ij} = m_{i1}m_{1j}$ yields

$$a_1 b_1 m_{ij} = a_i b_1 a_1 b_j$$

As $a_1, b_1 \neq 0$ and R is a commutative UFD (and, in particular, it is integral), $m_{ij} = a_i b_j$ for every $i, j > 1$. As the latter holds also for $i = 1$ or $j = 1$ by construction, then letting $A = (a_i)$ and $B = (b_j)$ yields $M = AB$ as desired. \square

Lemma 4.5.2. *If R is a field and $M = (m_{ij}) \in \text{Mat}_{n \times m}(R)$ is a nonzero matrix such that all the 2×2 minors have zero determinant, then the decomposition $M = AB$ stated by the previous lemma is unique in the sense that if $M = AB = A'B'$ for some matrices $A = (a_i), A' = (a'_i) \in \text{Mat}_{n \times 1}(R)$ and $B = (b_i), B' = (b'_i) \in \text{Mat}_{1 \times m}(R)$ then there exists a nonzero $\rho \in R$ such that $A' = \rho A$ and $B' = \rho^{-1} B$.*

Proof. Let m_{ij} be a nonzero element of M . Then the i -th row of M is nonzero and we have

$$a_i B = a'_i B'$$

with $a_i \neq 0$ and $a'_i \neq 0$. Then a'_i is invertible and we get that $B' = \frac{a_i}{a'_i} B$. Similarly, as the j -th column of M is nonzero we get

$$A b_j = A' b'_j$$

with $b_j \neq 0$ and $b'_j \neq 0$. Then b'_j is invertible and we get that $A' = \frac{b_j}{b'_j} A$. Finally, let $\rho = \frac{b_j}{b'_j}$ and note that as $m_{ij} = a_i b_j = a'_i b'_j \neq 0$, one gets

$$\frac{a_i}{a'_i} = \frac{m_{ij}/b_j}{m_{ij}/b'_j} = \frac{b'_j}{b_j} = \rho^{-1}$$

\square

Remark 4.5.3. *If $n = m$, then we can rewrite the nullity condition for the minors of M in a more compact way. For any matrix $M \in \text{Mat}_{n \times n}(R)$, all the 2×2 minors of M have null determinant in R if and only if*

$$\wedge^2 M = 0$$

We will introduce some notations that will be useful in order to work with linear morphisms between algebras of matrices.

Let us consider a bijection $\sigma : [1, n] \times [1, m] \rightarrow [1, n'] \times [1, m']$. Abusing the notation, let

$$\sigma : \text{Mat}_{n \times m}(R) \longrightarrow \text{Mat}_{n' \times m'}(R)$$

be the isomorphism that sends a matrix $M = (m_{ij}) \in \text{Mat}_{n \times m}(R)$ to the $n' \times m'$ matrix whose entry (i, j) is

$$(\sigma(M))_{ij} = m_{\sigma^{-1}(i,j)}$$

In particular, given a bijection $\tau : [1, n] \times [1, m] \rightarrow [1, nm] \times \{1\} = [1, nm]$ and a matrix $M \in \text{Mat}_{n \times m}(R)$, $\tau(M) \in \text{Mat}_{nm \times 1}(R) \cong R^{nm}$ is the column vector

obtained by placing all entries of M in a column using the bijection τ . Reciprocally, given such vector $V \in \text{Mat}_{nm \times 1}(R) \cong R^{nm}$, then $\tau^{-1}(V)$ is the corresponding matrix.

In order to simplify the notation, from now on, let us fix once and for all the bijection $\tau : [1, r]^2 \rightarrow [1, r^2]$ that places the entries of the matrix in row order, i.e.

$$\tau(i, j) = (i - 1)r + j$$

We will also fix the bijection $\iota : [1, r]^2 \rightarrow [1, r]^2$ sending $\iota(i, j) = (j, i)$, so that for every matrix $M \in \text{Mat}_{n \times n}(R)$

$$\iota(M) = M^t$$

Lemma 4.5.4. *Let R be a UFD. For every $n > 0$ there exists a bijection*

$$\sigma : [1, n^2]^2 \times [1, n^2]^2$$

such that given any matrix $M \in \text{GL}(\text{Mat}_{n \times n}(R)) \xrightarrow{\tau} \text{GL}_{n^2}(R)$, M is the matrix associated to a linear transformation of the form

$$X \mapsto AXB$$

for some $A, B \in \text{Mat}_{n \times n}(R)$ if and only if

$$\wedge^2(\sigma(M)) = 0$$

In that case, we will denote $M = \mathcal{M}_{A,B}$

Proof. The matrix $M \in \text{GL}_{n^2}(R)$ induced by the given linear transformation is the given by

$$\begin{array}{ccc} R^{n^2} & \longrightarrow & R^{n^2} \\ V & \longmapsto & \tau(A\tau^{-1}(V)B) \end{array}$$

For the bijection τ chosen above, it is straightforward to see that

$$M = A \otimes B^t$$

One just has to check that the morphisms

$$\text{End}(R^n) \cong (R^n)^* \otimes R^n \longrightarrow \text{End}(R^n) \cong (R^n)^* \otimes R^n$$

obtained by composing on the left with $A \in \text{End}(R^n)$ or on the right with $B \in \text{End}(R^n)$ correspond to

$$\text{Id} \otimes A : (R^n)^* \otimes R^n \longrightarrow (R^n)^* \otimes R^n$$

and

$$B^t \otimes \text{Id} : (R^n)^* \otimes R^n \longrightarrow (R^n)^* \otimes R^n$$

respectively, so the morphism represented by M is just $B^t \otimes A$. In order to write the matrix for the morphism, we need to select a basis for $(R^n)^* \otimes R^n$. The choice

of τ corresponds to selecting the basis of $(R^n)^* \otimes R$ in row order, so the matrix M in the basis induced by the isomorphism τ is $A \otimes B^t$.

By definition of tensor product, the entries of the matrix $A \otimes B^t$ are all the possible products $a_{ij}b_{kl}$ of an entry a_{ij} of A and an entry b_{kl} of B in a fixed order depending only on the dimension n . Therefore, there exists a fixed bijection $\sigma : [1, n^2]^2 \times [1, n^2]^2$ such that

$$\sigma(A \otimes B^t) = \tau(A) \cdot (\tau(B))^t$$

Therefore, the set of matrices $M \in \text{GL}_{n^2}(R)$ for which there exist $A, B \in \text{Mat}_{n \times n}(R)$ such that

$$M(V) = \tau(A\tau^{-1}(V)B)$$

is the set of matrices M such that there exist vectors $\tau(A), \tau(B) \in R^{n^2}$ such that

$$\sigma(M) = \tau(A) \cdot (\tau(B))^t$$

By Lemma 4.5.1, such vectors exist if and only if

$$\wedge^2(\sigma(M)) = 0$$

□

Corollary 4.5.5. *Let R be a UFD and let σ be the bijection given by the previous lemma. Then $M = (m_{\alpha, \beta}) \in \text{GL}_{n^2}(R)$ is the matrix of an inner transformation*

$$X \mapsto AXA^{-1}$$

for some $A \in \text{GL}_n(R)$ if and only if $\wedge^2(\sigma(M)) = 0$ and for every $i, j \in [1, n]$

$$\sum_{k=1}^r m_{\sigma^{-1}(\tau(i,k), \tau(k,j))} = \sum_{k=1}^r m_{\sigma^{-1}(\tau(j,k), \tau(k,i))} = \delta_{ij}$$

Proof. By the lemma, if $\wedge^2(\sigma(M)) = 0$ then there exist matrices $A, B \in \text{Mat}_{n \times n}(R)$ such that M is the map induced by

$$X \mapsto AXB$$

then M is an inner transformation if and only if A and B are inverses, i.e., if and only if $AB = BA = I$, where I is the identity matrix. This holds if and only if for every $i, j = 1, \dots, n$

$$\sum_{k=1}^n a_{ik}b_{kj} = \sum_{k=1}^r b_{ik}a_{kj} = \delta_{ij} \quad (4.5.1)$$

On the other hand, as

$$\sigma(M) = \tau(A) \cdot (\tau(B))^t$$

then for every $i, j, k, l = 1, \dots, n$

$$a_{ij}b_{kl} = m_{\sigma^{-1}(\tau(i,j), \tau(k,l))}$$

so equality (4.5.1) holds if and only if

$$\sum_{k=1}^r m_{\sigma^{-1}(\tau(i,k),\tau(k,j))} = \sum_{k=1}^r m_{\sigma^{-1}(\tau(j,k),\tau(k,i))} = \delta_{ij}$$

Reciprocally, if M is an inner transformation, $\wedge^2(\sigma(M)) = 0$ and

$$\sigma(M) = \tau(A) \cdot (\tau(A^{-1}))^t$$

so for every $i, j, k, l = 1, \dots, n$

$$a_{ij}(A^{-1})_{kl} = m_{\sigma^{-1}(\tau(i,j),\tau(k,l))}$$

Then the corollary follows from

$$\sum_{k=1}^n a_{ik}(A^{-1})_{kj} = \sum_{k=1}^r (A^{-1})_{ik} a_{kj} = \delta_{ij}$$

□

Note that if R is a field, Lemma 4.5.2 implies that if M is the matrix of an inner transformation, then the matrix A is uniquely determined up to product by nonzero elements of R .

Now let R be a local principal ideal domain which is not a field (i.e., a discrete valuation ring). For example, within the scope of this article, the following Lemmas will be applied to the local ring of a smooth complex projective curve $R = \mathcal{O}_{X,x}$. Let \mathfrak{m} be the maximal ideal in R and let $K = \text{Frac}(R)$ be the field of fractions. As R is a principal domain, $\mathfrak{m} = (z)$ for some $z \in R$. We will denote by

$$\nu_z : K \rightarrow \mathbb{Z}$$

the single discrete valuation on K extending the canonical z -valuation of the elements in R , i.e., the only possible discrete valuation for which $R = \{a \in K : \nu_z(a) \geq 0\}$. Let $\text{PEnd}_n(R) \subset \text{Mat}_{n \times n}(R)$ be the R -module of $n \times n$ matrices whose elements below the diagonal are multiples of z , i.e., the R -module consisting of matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ za_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ za_{r1} & za_{r2} & \cdots & a_{rr} \end{pmatrix}$$

where $a_{ij} \in R$. It is clear that $\text{PEnd}_n(R)$ forms a sub R -algebra of $\text{Mat}_{n \times n}(R)$. If we suppose that $z \neq 0$ (i.e., that R is not a field), then as an R -module, $\text{PEnd}(R)$ is isomorphic to $\text{Mat}_{n \times n}(R)$, but they are not isomorphic as R -algebras.

Later on we will have to work with this kind of isomorphisms with a little more generality, so it is convenient to fix some general notation. Let us consider a formal sum of indexes in $[1, n] \times [1, m]$

$$\Xi = \sum_{(i,j) \in [1,n] \times [1,m]} \Xi_{ij} \cdot (i, j) \in \mathbb{Z}([1, n] \times [1, m])$$

Then we denote by $\mathcal{Z}_\Xi : \text{Mat}_{n \times m}(K) \cong \text{Mat}_{n \times m}(K)$ the isomorphism of K -modules that sends a matrix $M = (m_{ij})$ to the matrix $\mathcal{Z}_\Xi(M)$ whose element (i, j) is

$$\mathcal{Z}_\Xi(M)_{ij} = z^{\Xi_{ij}} m_{ij}$$

From the definition, it is clear that

$$\begin{aligned} \mathcal{Z} &: \mathbb{Z}([1, n] \times [1, m]) \longrightarrow \text{GL}(\text{Mat}_{n \times m}(K)) \\ \Xi &\longmapsto \mathcal{Z}_\Xi \end{aligned}$$

is a group homomorphism.

Let $\Xi_T = \sum_{1 \leq j < i \leq n} (i, j)$ be the sum of indexes below the diagonal. Then it is clear that the restriction of $\mathcal{Z}_{\Xi_T} : \text{Mat}_{n \times n}(K) \rightarrow \text{Mat}_{n \times n}(K)$ to $\text{Mat}_{n \times n}(R)$ is precisely the isomorphism

$$\mathcal{Z}_{\Xi_T} : \text{Mat}_{n \times n}(R) \cong \text{PEnd}_n(R)$$

Using the isomorphism $\tau : \text{Mat}_{n \times n}(K) \cong K^{n^2}$ we can compute the matrix Z_Ξ for the isomorphism $\tau \circ \mathcal{Z}_\Xi \circ \tau^{-1}$. For every $V \in K^{n^2}$ let

$$V_\Xi = \mathcal{Z}_\Xi V = \tau(\mathcal{Z}_\Xi(\tau^{-1}(V)))$$

Then, by definition of \mathcal{Z}_Ξ , if $V_\Xi = (v_{\Xi, i})_i$ then

$$v_{\Xi, i} = z^{\Xi_{\tau^{-1}(i)}} v_i$$

Given a bijection $\sigma : [1, n] \times [1, m] \rightarrow [1, n'] \times [1, m']$, let us denote

$$\sigma(\Xi) = \sum_{i, j} \Xi_{ij} \sigma(i, j) \in \mathbb{Z}[1, n'] \times [1, m']$$

Then, $\tau \circ \mathcal{Z}_\Xi \circ \tau^{-1} = \mathcal{Z}_{\sigma(\Xi)}$ and the matrix Z_Ξ is the diagonal matrix

$$Z_\Xi = \text{diag} \left(z^{\Xi_{\tau^{-1}(i)}} \right)$$

Lemma 4.5.6. *Let R be a local principal ideal domain which is not a field. Let $n > 0$ and let σ be the bijection given in Lemma 4.5.4. There exists a formal sum of indexes*

$$\Xi = \sum \Xi_{ij}(i, j) \in \mathbb{Z}[1, n^2]^2$$

with $-1 \leq \Xi_{ij} \leq 1$ such that given any matrix $M \in \text{GL}_{n^2}(R) \cong \text{GL}(\text{PEnd}_n(R))$, M is the matrix associated to a linear transformation of the form

$$X \mapsto AXB$$

for some $A, B \in \text{Mat}_{n \times n}(K)$ if and only if

$$\wedge^2(\sigma(\mathcal{Z}_{-\Xi}(M))) = 0$$

Moreover, if $\mathcal{Z}_{-\Xi}(M) \in \text{Mat}_{n^2 \times n^2}(R)$, then A and B can be chosen in $\text{Mat}_{n \times n}(R)$.

Proof. Let $M \in \mathrm{GL}_{n^2}(R)$ be the matrix associated to a map $X \mapsto AXB$. Then it sends a vector $V \in R^{n^2}$ to

$$MV = \tau \left(\mathcal{Z}_{-\Xi_T} (A \mathcal{Z}_{\Xi_T} (\tau^{-1}(V)) B) \right)$$

Then we can view M as the restriction to R^{n^2} of the composition of the following morphisms

$$\begin{array}{ccc} K^{n^2} & \xrightarrow{M} & K^{n^2} \\ \tau \circ \mathcal{Z}_{\Xi_T} \circ \tau^{-1} \downarrow & & \uparrow \tau \circ \mathcal{Z}_{-\Xi_T} \circ \tau^{-1} \\ K^{n^2} & \xrightarrow{\mathcal{M}_{A,B}} & K^{n^2} \end{array}$$

By the computations carried away in the previous lemmata, the matrix M is the product

$$M = \mathcal{Z}_{-\Xi_T} (A \otimes B^t) \mathcal{Z}_{\Xi_T}$$

We will see that then there exists a formal sum of indexes $\Xi \in \mathbb{Z}[1, n^2]^2$ such that

$$M = \mathcal{Z}_{\Xi} (A \otimes B^t)$$

For any $\Xi \in \mathbb{Z}[1, n]^2$, taking the product on the left by $Z_{\Xi} = \mathrm{diag}(z^{\Xi_{\tau^{-1}(i)}})$ is equivalent to multiplying the i -th row of the matrix by $z^{\Xi_{\tau^{-1}(i)}}$ for each $i = 1, \dots, n^2$, so if we set

$$\Xi_l = \sum_{i,j=1}^{n^2} \Xi_{\tau^{-1}(i)}(i, j)$$

for every matrix $N \in \mathrm{Mat}_{n^2 \times n^2}(K)$

$$Z_{\Xi} N = \mathcal{Z}_{\Xi_l}(N)$$

Similarly, product on the right by Z_{Ξ} is equivalent to multiplying the i -th column of the matrix by $z^{\Xi_{\tau^{-1}(i)}}$ for each $i = 1, \dots, n^2$, so if we set

$$\Xi_r = \sum_{i,j=1}^{n^2} \Xi_{\tau^{-1}(j)}(i, j)$$

for every matrix $N \in \mathrm{Mat}_{n^2 \times n^2}(K)$ yields

$$N Z_{\Xi} = \mathcal{Z}_{\Xi_r}(N)$$

Therefore, setting

$$\Xi = -(\Xi_T)_l + (\Xi_T)_r$$

we conclude that

$$M = \mathcal{Z}_{\Xi} (A \otimes B^t)$$

Let us check that $-1 \leq \Xi_{\alpha,\beta} \leq 1$. For each $(\alpha, \beta) = (\tau(i, j), \tau(k, l))$ yields

$$-(\Xi_T)_l)_{\alpha,\beta} = -(\Xi_T)_{i,j} = \begin{cases} -1 & j < i \\ 0 & j \geq i \end{cases}$$

$$((\Xi_T)_r)_{\alpha,\beta} = (\Xi_T)_{k,l} = \begin{cases} 1 & l < k \\ 0 & l \geq k \end{cases}$$

So it yields $-1 \leq \Xi_{\alpha,\beta} \leq 1$. As an example, we show the matrix representing Ξ for $n = 4$

$$\left(\begin{array}{cccc|cccc|cccc|cccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ \hline -1 & -1 & -1 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ \hline -1 & -1 & -1 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 \\ -1 & -1 & -1 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ \hline -1 & -1 & -1 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 \\ -1 & -1 & -1 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 \\ -1 & -1 & -1 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \end{array} \right)$$

Given a matrix $M \in \text{GL}_{n^2}(R)$, in general $\mathcal{Z}_{-\Xi}(M) \in \text{Mat}_{n^2 \times n^2}(K)$. Following the proof of Lemma 4.5.4, there exist matrices $A, B \in \text{Mat}_{n \times n}(K)$ such that $\mathcal{Z}_{-\Xi}(M) = A \otimes B^t$ if and only if

$$\wedge^2(\sigma(\mathcal{Z}_{-\Xi}(M))) = 0$$

moreover, if $\mathcal{Z}_{-\Xi}(M) \in \text{Mat}_{n^2 \times n^2}(R)$, then as R is a principal ideal domain then if

$$\wedge^2(\sigma(\mathcal{Z}_{-\Xi}(M))) = 0$$

there exist $A, B \in \text{Mat}_{n \times n}(R)$ such that $\mathcal{Z}_{-\Xi}(M) = A \otimes B^t$. \square

Similarly to the non-parabolic case, we will denote by

$$\mathcal{M}_{A,B}^{\text{par}} \in \text{Mat}_{n^2 \times n^2}(K)$$

the matrix associated to a map $X \mapsto AXB$ for $A, B \in \text{Mat}_{n \times n}(K)$. More explicitly, for every $V \in K^{n^2}$, let

$$\mathcal{M}_{A,B}^{\text{par}} V = \tau(\mathcal{Z}_{-\Xi_T}(A \mathcal{Z}_{\Xi_T}(\tau^{-1}(V)) B))$$

Note that, in general, if $A, B \in \text{Mat}_{n \times n}(K)$, $\mathcal{M}_{A,B}^{\text{par}} V \in K^{n^2}$ even if $V \in R^{n^2}$. If $\mathcal{M}_{A,B}^{\text{par}} \in \text{GL}_{n^2}^2(R)$, then this imposes some conditions on the structure of A and B .

Lemma 4.5.7. *If $M = \mathcal{M}_{A,B}^{\text{par}} = \mathcal{M}_{A',B'}^{\text{par}}$ is a nonzero matrix for some $A, A', B, B' \in \text{Mat}_{n \times n}(K)$, then there exists a nonzero $\rho \in K$ such that $A' = \rho A$ and $B' = \rho^{-1} B$.*

Proof. From the previous lemma, yields

$$\sigma(\mathcal{Z}_{-\Xi'}(M)) = \tau(A) \cdot \tau(B)^t = \tau(A') \cdot \tau(B)^t$$

and now we apply Lemma 4.5.2. \square

Lemma 4.5.8. *Suppose that there exist matrices $A, B \in \text{Mat}_{n \times n}(K)$ such that $M = \mathcal{M}_{A,B}^{\text{par}} \in \text{GL}_{n^2}(R)$. Then there exist $A', B' \in \text{Mat}_{n \times n}(R)$ such that*

$$M = \mathcal{M}_{A,B/z}^{\text{par}} = \mathcal{M}_{A',B'/z}^{\text{par}}$$

Proof. By the Lemma 4.5.6, $\wedge^2(\sigma(\mathcal{Z}_{-\Xi}(M))) = 0$. Then $\wedge^2(\sigma(\mathcal{Z}_{-\Xi}(zM))) = 0$. As $-1 \leq \Xi_{\alpha\beta} \leq 1$ for all $\alpha, \beta = 1, \dots, n^2$, then $\mathcal{Z}_{-\Xi}(zM) \in \text{Mat}_{n^2 \times n^2}(R)$. Therefore, there exist $A', B' \in \text{Mat}_{n \times n}(R)$ such that zM is the matrix $\mathcal{M}_{A',B'}^{\text{par}}$. The result yields dividing the matrix by z . \square

Corollary 4.5.9. *Let $A \in \text{GL}_n(K)$ be a matrix such that $\mathcal{M}_{A,A^{-1}}^{\text{par}} \in \text{GL}_{n^2}(R)$. Then, there exist nonzero matrices $A', B' \in \text{Mat}_{n \times n}(R)$ such that B'/z is the inverse of A' in $\text{GL}_n(K)$*

$$\mathcal{M}_{A,A^{-1}}^{\text{par}} = \mathcal{M}_{A',B'/z}^{\text{par}}$$

Proof. By the previous lemma, there exist nonzero $A', B' \in \text{GL}_{n^2}(R)$ such that

$$\mathcal{M}_{A,A^{-1}}^{\text{par}} = \mathcal{M}_{A',B'/z}^{\text{par}}$$

Now, we apply the corollary 4.5.5, to

$$\mathcal{M}_{A,A^{-1}} = \mathcal{Z}_{-\Xi}(\mathcal{M}_{A,A^{-1}}^{\text{par}}) = \mathcal{Z}_{-\Xi}(\mathcal{M}_{A',B'/z}^{\text{par}}) = \mathcal{M}_{A',B'/z}$$

\square

Lemma 4.5.10. *Suppose that there exists a matrix $A \in \text{GL}_n(R)$ such that $\mathcal{M}_{A,A^{-1}}^{\text{par}} \in \text{GL}_{n^2}(R)$. Then $A \in \text{PEnd}_n(R) \cap \text{GL}_n(R)$.*

Proof. As $\det(A)$ is invertible

$$A^{-1} = \det(A)^{-1} \text{ad}(A)^t$$

Let us denote by A_{ij} the (i, j) adjoint of matrix A , i.e., the determinant of the complement minor to the element (i, j) taken with the corresponding sign. Let $M = \mathcal{M}_{A,A^{-1}}^{\text{par}}$. Then

$$M = \mathcal{Z}_{\Xi}(A \otimes (A^{-1})^t) = \det(A)^{-1} \mathcal{Z}_{\Xi}(A \otimes \text{ad}(A)) \in \text{GL}_n(R)$$

Looking at the blocks of $A \otimes \text{ad}(A)$ below the diagonal, $\mathcal{Z}_{\Xi}(A \otimes \text{ad}(A))$ being a matrix in R implies that

$$z|a_{ij}A_{kl}$$

for $j < i$, $k < i$ and $j \leq l$. In particular, this implies that $z|a_{ij}A_{kl}$ for $k \leq i-1 < l$ and every $j < i$. Let us prove that this implies that $z|a_{ij}$ for $j < i$, so that

$A \in \text{PEnd}(R)$. Suppose that $z \nmid a_{ij}$ for some $j < i$. Then $z \mid A_{kl}$ for all $k \leq i-1 < l$. Then we will prove that

$$z \mid \det((A_{kl})_{k,l=1}^n) = \det(\text{ad}(A)) = \det(A)^n \det(A^{-1})$$

which would let to contradiction, as $A \in \text{GL}_n(R)$ and z is not invertible in R . Let us prove it by induction on n . For $n = 1$ the statement is trivial. Suppose that it is true for $n' < n$. If $i = 2$, then $z \mid A_{1l}$ for every l , so $(A_{kl})_{k,l=1}^n$ has a row full of multiples of z and, therefore, its determinant is a multiple of z . If $i > 2$, let us develop the determinant of (A_{kl}) through the first row

$$\det((A_{kl})_{k,l=1}^n) = \sum_{l=1}^n (-1)^{l+1} A_{1l} \det(D^{1l})$$

where D^{kl} is the complement minor of (A_{kl}) for the element (k, l) . For $l \geq i$, $z \mid A_{1l}$, so it is enough to prove that $z \mid \det(D^{1l})$ for $l < i$. D^{kl} is obtained by removing the first row and the l -th column of $(A_{kl})_{k,l=1}^n$. As $l < i$, D^{kl} contains all the elements A^{kl} for $1 < k \leq i-1 < l$ in the positions $k' = k-1$, $l' = l-1$, so we know that $z \mid (D^{kl})_{k',l'}$ for $k' \leq i-2 < l'$. Now, we apply the induction hypothesis to the $(n-1)$ -dimensional matrix D^{kl} . \square

Lemma 4.5.11. *Let $M \in \text{GL}_{n^2}(R)$ be a matrix such that there exists $A \in \text{GL}_n(K)$ satisfying $M = \mathcal{M}_{A, A^{-1}}^{\text{par}}$. Then, there exists a matrix $A' \in \text{PEnd}_n(R) \cap \text{GL}_n(R)$ and an integer $0 \leq k < n$ such that*

$$M = \mathcal{M}_{(A'H^k), (A'H^k)^{-1}}^{\text{par}}$$

where $H \in \text{GL}_n(K)$ is the matrix

$$H = \left(\frac{0 \mid I_{n-1}}{z \mid 0} \right) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ z & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Proof. First, let us prove that $\mathcal{M}_{H, H^{-1}}^{\text{par}} \in \text{GL}_{n^2}(R)$. As

$$\det(\mathcal{M}_{H, H^{-1}}^{\text{par}}) = \det(Z_{\Xi_T}) \det(H) \det(H^{-1}) \det(Z_{\Xi_T}^{-1}) = 1$$

it is enough to prove that $\mathcal{M}_{H, H^{-1}}^{\text{par}} \in \text{GL}_{n^2}(R)$. We can easily compute that

$$H' := (H^{-1})^t = \left(\frac{0 \mid I_{n-1}}{z^{-1} \mid 0} \right) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ z^{-1} & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Now it is enough to prove that $\mathcal{M}_{H,H^{-1}} = \mathcal{Z}_{-(\Xi_T)_l+(\Xi_T)_r}(H \otimes H') \in \text{Mat}_{n^2 \times n^2}(R)$, but it is straightforward to check that

$$\mathcal{Z}_{-(\Xi_T)_l+(\Xi_T)_r}(H \otimes H') = \left(\begin{array}{c|c} 0 & I_{n-1} \\ \hline 1 & 0 \end{array} \right) \otimes \left(\begin{array}{c|c} 0 & I_{n-1} \\ \hline 1 & 0 \end{array} \right)$$

Observe that, as $H^n = zI$, yields $H^{-k} = H^{n-k}/z$, so

$$\mathcal{M}_{H^{-k},H^k}^{\text{par}} = \mathcal{M}_{H^{n-k}/z,H^{k-n}}^{\text{par}} = \mathcal{M}_{H^{n-k},H^{k-n}}^{\text{par}} = \left(\mathcal{M}_{H,H^{-1}}^{\text{par}} \right)^{n-k} \in \text{GL}_{n^2}(R)$$

Corollary 4.5.9 allows us to find matrices $A', B' \in \text{Mat}_{n \times n}(R)$ such that $M = \mathcal{M}_{A',B'}^{\text{par}}/z$ and $A'B'/z = B'A'/z = I$. First, let us prove that we can assume that $z^n \nmid \det(A')$. As $A'B'/z = I$, we get that

$$\det(A') \det(B') = z^n$$

As z is not invertible in R and $\det(B') \in R$, then $\det(A') \mid z^n$. Suppose that $z^n \mid \det(A')$. Then $z \nmid \det(B')$, so $\det(B') \notin \mathfrak{m}$ and therefore, $\det(B)$ is invertible in R . As the inverse of B' is A'/z , then

$$\frac{1}{z}A' = (B')^{-1} = \frac{1}{\det(B)} \text{ad}(B)^t$$

where, $\text{ad}(B)$ is the adjoint matrix of B . As the adjoint belongs to $\text{Mat}_{n \times n}(R)$ and $\det(B)^{-1} \in R$, then $\frac{A'}{z} \in \text{Mat}_{n \times n}(R)$. Then, $M = \mathcal{M}_{A'/z, (A'/z)^{-1}}^{\text{par}}$ and $A'/z \in \text{GL}_n(R)$.

If $z \nmid \det(A')$, then $\det(A')$ is invertible and, thus, $A' \in \text{GL}_n(R)$, so $M = \mathcal{M}_{A', (A')^{-1}}^{\text{par}}$. Now suppose that $z^k \mid \det(A')$ but $z^{k+1} \nmid \det(A')$ for some $0 < k < n$. Then

$$M' = \mathcal{M}_{A',B'/z}^{\text{par}} \mathcal{M}_{H^{-k},H^k}^{\text{par}} = \mathcal{M}_{A'H^{-k},H^k B'/z}^{\text{par}} \in \text{GL}_{n^2}(R)$$

so there exist matrices $A'', B'' \in \text{Mat}_{n \times n}(R)$ with $z^n \nmid A''$ and $(A'')^{-1} = B''/z$ such that

$$\mathcal{M}_{A'H^{-k},H^k B'/z}^{\text{par}} = M' = \mathcal{M}_{A'',B''/z}^{\text{par}}$$

but then, by Lemma 4.5.7 there exists a nonzero $\rho \in K$ such that

$$A'' = \rho A' H^{-k}$$

Taking determinants

$$\det(A'') = \rho^n \frac{\det(A')}{z^k}$$

We have $z^{-k} \det(A') \notin \mathfrak{m}$ by hypothesis and $z^n \nmid \det(A'') \in R$. Taking the z -valuation ν_z at both sides yields

$$\nu_z(\det(A'')) = n\nu_z(\rho)$$

As $0 \leq \nu_z(\det(A'')) < n$, yields $\nu_z(\rho) = 0$, so ρ is invertible in R and we get

$$A' = \rho^{-1} A'' H^k$$

Moreover, $\nu_z(\det(A'')) = 0$, so $\det(A'')$ is invertible, and therefore, $\rho^{-1} A'' \in \text{GL}_n(R)$. From Lemma 4.5.10, $\rho^{-1} A'' \in \text{PEnd}_n(R) \cap \text{GL}_n(R)$ and the Lemma follows. \square

Lemma 4.5.12. *Let $\{U_\alpha\}_{\alpha \in I}$ be a good cover of (X, D) such that for every $x \in D$ there exists a unique $\alpha_x \in I$ such that $x \in U_{\alpha_x}$. Let (E, E_\bullet) be a parabolic vector bundle described by a cocycle $\varphi_{\alpha\beta} : U_{\alpha\beta} \rightarrow \mathcal{G}|_{U_{\alpha\beta}}$. Then $\mathcal{H}_x(E, E_\bullet)$ is described by the following cocycle $\psi_{\alpha\beta} : U_{\alpha\beta} \rightarrow \mathcal{G}|_{U_{\alpha\beta}}$*

$$\psi_{\alpha\beta} = \begin{cases} \varphi_{\alpha\beta} & x \notin U_\alpha \cup U_\beta \\ H\varphi_{\alpha\beta} & x \in U_\beta \\ \varphi_{\alpha\beta}H^{-1} & x \in U_\alpha \end{cases}$$

where $H = \left(\begin{array}{c|c} 0 & I_{n-1} \\ \hline z & 0 \end{array} \right)$ and z is a local coordinate in U_{α_x} centered in x .

Proof. First, let us prove that ψ is a cocycle. Let $\alpha, \beta, \gamma \in I$ with $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$ and let us compute $\psi_{\gamma\alpha}\psi_{\beta\gamma}\psi_{\alpha\beta}$. If x does not belong to any of the open sets, ψ coincides with φ and

$$\psi_{\gamma\alpha}\psi_{\beta\gamma}\psi_{\alpha\beta} = \varphi_{\gamma\alpha}\varphi_{\beta\gamma}\varphi_{\alpha\beta} = 1$$

If $x \in U_\alpha$

$$\psi_{\gamma\alpha}\psi_{\beta\gamma}\psi_{\alpha\beta} = H\varphi_{\gamma\alpha}\varphi_{\beta\gamma}\varphi_{\alpha\beta}H^{-1} = HH^{-1} = 1$$

If $x \in U_\beta$

$$\psi_{\gamma\alpha}\psi_{\beta\gamma}\psi_{\alpha\beta} = \varphi_{\gamma\alpha}\varphi_{\beta\gamma}H^{-1}H\varphi_{\alpha\beta} = \varphi_{\gamma\alpha}\varphi_{\beta\gamma}\varphi_{\alpha\beta} = 1$$

and if $x \in U_\gamma$

$$\psi_{\gamma\alpha}\psi_{\beta\gamma}\psi_{\alpha\beta} = \varphi_{\gamma\alpha}H^{-1}H\varphi_{\beta\gamma}\varphi_{\alpha\beta} = \varphi_{\gamma\alpha}\varphi_{\beta\gamma}\varphi_{\alpha\beta} = 1$$

Recall that $E_x^2 \subset E$ denotes the second step of the filtration by subsheaves defining the parabolic structure of (E, E_\bullet) at x and it is precisely the underlying vector bundle of $\mathcal{H}_x(E, E_\bullet)$ (see section 4.4). The trivialization induced by $\varphi_{\alpha\beta}$ at the stalk E_x is precisely

$$(E_x^2)_x \cong \mathfrak{m} \oplus \mathcal{O}_{X,x}^{r-1} \subset \mathcal{O}_{X,x}^r \xrightarrow{\varphi} E_x$$

A trivialization of $E_x^2 \cong \mathcal{H}_x^{E_x, 2}(E)$ compatible with the induced parabolic structure would be the one obtained by “rotating” the given one through the procedure described in the previous chapter

$$(E_x^2)_x \xrightarrow{\psi} \mathcal{O}_{X,x}^r \xrightarrow{\Xi_r} \mathcal{O}_{X,x}^{r-1} \oplus \mathfrak{m} \xrightarrow{\pi} \mathfrak{m} \oplus \mathcal{O}_{X,x}^{r-1}$$

where π is the permutation sending $\pi(i) = i - 1$ for $i > 1$ and $\pi(1) = r$. Therefore, we get

$$(E_x^2)_x \xrightarrow{H} \mathfrak{m} \oplus \mathcal{O}_{X,x}^{r-1} \subset \mathcal{O}_{X,x}^r \xrightarrow{\varphi} E_x$$

and we are done. \square

Corollary 4.5.13. *Let (E, E_\bullet) be a parabolic vector bundle. Then $\text{PEnd}_0(E, E_\bullet)$ is isomorphic to $\text{PEnd}_0(\mathcal{H}_x(E, E_\bullet))$ as a Lie algebra bundle and at the stalk at the parabolic point $x \in D$ the isomorphism coincides with*

$$\mathcal{M}_{H, H^{-1}}^{\text{par}} : \text{PEnd}_0(E, E_\bullet)_x \cong \text{PEnd}_0(\mathcal{H}_x(E, E_\bullet))_x$$

Lemma 4.5.14. *Let (E, E_\bullet) and (E', E'_\bullet) be parabolic vector bundles of rank r such that $\mathrm{PEnd}_0(E, E_\bullet)$ and $\mathrm{PEnd}_0(E', E'_\bullet)$ are isomorphic as Lie algebra bundles. Then (E', E'_\bullet) can be obtained from (E, E_\bullet) through a combination of the following transformations*

1. *Tensorization with a line bundle over X , $(E, E_\bullet) \mapsto (E \otimes L, E_\bullet \otimes L)$*
2. *Parabolic dualization $(E, E_\bullet) \mapsto (E, E_\bullet)^\vee$*
3. *Hecke transformation at a parabolic point $x \in D$, $(E, E_\bullet) \mapsto \mathcal{H}_x(E, E_\bullet)$.*

Proof. Giving a vector bundle $\mathrm{PEnd}_0(E)$ with its Lie algebra structure is equivalent to giving an $\mathrm{Aut}(\mathfrak{par}\mathfrak{sl})$ -torsor $P_{\mathrm{Aut}(\mathfrak{par}\mathfrak{sl})}$ which admits a reduction to a \mathcal{G} -torsor (which corresponds to the parabolic vector bundle (E, E_\bullet)). We will analyze the possible reductions from a given $\mathrm{Aut}(\mathfrak{par}\mathfrak{sl})$ -torsor in two steps.

$$\mathcal{G} \rightarrow \mathrm{Inn}(\mathfrak{par}\mathfrak{sl}) \hookrightarrow \mathrm{Aut}(\mathfrak{par}\mathfrak{sl})$$

First, note that there is an exact sequence of sheaves of groups

$$1 \longrightarrow \mathrm{Inn}(\mathfrak{par}\mathfrak{sl}) \longrightarrow \mathrm{Aut}(\mathfrak{par}\mathfrak{sl}) \longrightarrow \mathrm{Out}(\mathfrak{par}\mathfrak{sl}) \longrightarrow 1$$

Our first step is to compute the outer automorphisms of $\mathfrak{par}\mathfrak{sl}$. Over a non-parabolic point $x \notin D$, taking stalks the previous short exact sequence simply reduces to

$$1 \longrightarrow \mathrm{Inn}(\mathfrak{sl}) = \mathrm{PGL}_r \longrightarrow \mathrm{Aut}(\mathfrak{sl}) \longrightarrow \mathrm{Out}(\mathfrak{sl}) = \mathbb{Z}_2 \longrightarrow 1$$

Therefore, in order to determine $\mathrm{Out}(\mathfrak{par}\mathfrak{sl})$, we only need to determine the stalk of $\mathrm{Out}(\mathfrak{par}\mathfrak{sl})$ at a parabolic point. The single nontrivial outer automorphism of \mathfrak{sl} is the one induced by duality of the underlying vector space. Given a parabolic full flag vector bundle (E, E_\bullet) , parabolic duality induces an outer isomorphism of the algebra $\mathfrak{par}\mathfrak{sl}$ extending the previous one over non-parabolic points. Let $x \in D$. Let $o_1, o_2 \in \mathrm{Out}(\mathfrak{par}\mathfrak{sl})_x$ be two germs of sections at the parabolic point. Composing with the dualization action if necessary, we may assume that o_1 and o_2 coincide generically. Then there exist germs of sections $\bar{o}_1, \bar{o}_2 \in \mathrm{Aut}(\mathfrak{par}\mathfrak{sl})_x$ such that $s := \bar{o}_1 \circ \bar{o}_2^{-1} \in \mathrm{Aut}(\mathfrak{par}\mathfrak{sl})_x$ is a germ whose restriction to the open set correspond to an inner automorphism.

Let $\mathcal{O}_{X,x}$ be the stalk of the structure sheaf at $x \in D$. Let \mathfrak{m} be the maximal ideal in $\mathcal{O}_{X,x}$ and let $\mathcal{K} = \mathcal{O}_{X,x}/\mathfrak{m}$ be the field of fractions. As X is a smooth curve, $\mathcal{O}_{X,x}$ is a principal ideal domain, so $\mathfrak{m} = (z)$ for some germ $z \in \mathcal{O}_{X,x}$. Therefore, an element of $\mathfrak{par}\mathfrak{sl}_x$ is represented by an $r \times r$ matrix of elements of $\mathcal{O}_{X,x}$ whose elements below the diagonal are multiples of z , i.e., it is a matrix of the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ za_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ za_{r1} & za_{r2} & \cdots & a_{rr} \end{pmatrix}$$

where $a_{ij} \in \mathcal{O}_{X,x}$ and $\sum_{i=1}^r a_{ii} = 0$. The germ s is, in particular, a germ of $\mathrm{GL}(\mathfrak{par}\mathfrak{sl})$. Trace 0 matrices form a linear codimension 1 subspace of $\mathfrak{par}\mathfrak{gl}$ whose

complement is generated by the identity matrix. Therefore, any element of $\mathrm{GL}(\mathbf{par}\mathfrak{sl})$ extends to an element of $\mathrm{GL}(\mathbf{par}\mathfrak{gl})$ sending the identity matrix to itself. Moreover, if an element of $\mathrm{GL}(\mathbf{par}\mathfrak{sl})$ belongs to $\mathrm{Aut}(\mathbf{par}\mathfrak{sl})$, the extension belongs to $\mathrm{Aut}(\mathbf{par}\mathfrak{gl})$, as the identity matrix belongs to the kernel of the Lie bracket in $\mathbf{par}\mathfrak{gl}$.

Then, any germ $s \in \mathrm{Aut}(\mathbf{par}\mathfrak{sl})_x$ can be described by an invertible $r^2 \times r^2$ matrix of elements of $\mathcal{O}_{X,x}$ by embedding $\mathrm{Aut}(\mathbf{par}\mathfrak{sl})_x \hookrightarrow \mathrm{GL}(\mathrm{Mat}_{n \times n}(\mathcal{O}_{X,x})) \stackrel{\tau}{\cong} \mathrm{GL}_{r^2}(\mathcal{O}_{X,x})$. Let $S = (s_{ij}) \in \mathrm{GL}_{r^2}(\mathcal{O}_{X,x})$ be such matrix. As s corresponds generically to an inner automorphism, there exists a matrix $G \in \mathrm{GL}_{r^2}(\mathcal{K})$ such that $S = \mathcal{M}_{G,G^{-1}}^{\mathrm{par}}$. By Lemma 4.5.11, there exists a matrix $G \in \mathrm{PEnd}(\mathcal{O}_{X,x}) \cap \mathrm{GL}_n(\mathcal{O}_{X,x}) \cong \mathcal{G}_x$ and an integer $0 \leq k < r$ such that

$$S = \mathcal{M}_{GH^k, (GH^k)^{-1}}^{\mathrm{par}} = \mathcal{M}_{G,G^{-1}}^{\mathrm{par}} \circ \left(\mathcal{M}_{H,H^{-1}}^{\mathrm{par}} \right)^k$$

Moreover, as $\mathcal{M}_{H,H^{-1}}^{\mathrm{par}}$ is a conjugation operation in $\mathrm{Mat}_{r \times r}(\mathcal{K})$, it clearly preserves the 0-trace and it is a Lie algebra isomorphism. Therefore $\frac{\mathrm{Aut}(\mathbf{par}\mathfrak{sl})_x}{\mathbb{Z}_2 \times \mathrm{Inn}(\mathbf{par}\mathfrak{sl})_x}$ is generated by the order r automorphism induced from conjugation by the matrix H . One trivially checks that taking the dual and conjugating by H is the same as conjugating by H^{-1} and then taking the dual, so the outer automorphism group is

$$\mathrm{Out}(\mathbf{par}\mathfrak{sl})_x \cong \langle s, h \rangle / \{s^2 = 1, h^r = 1, sh = h^{-1}s\} = \mathbb{D}_r$$

where \mathbb{D}_r is the dihedral group of order r . Therefore, $\mathrm{Out}(\mathbf{par}\mathfrak{sl})_x$ fits in a sequence

$$1 \longrightarrow \mathbb{Z}_2 \times X \longrightarrow \mathrm{Out}(\mathbf{par}\mathfrak{sl}) \longrightarrow \mathbb{Z}_r \otimes \mathcal{O}_D \longrightarrow 0$$

The space of reductions of structure sheaf of $P_{\mathrm{Aut}(\mathbf{par}\mathfrak{sl})}$ to $\mathrm{Inn}(\mathbf{par}\mathfrak{sl})$ correspond to sections of the associated $\mathrm{Out}(\mathbf{par}\mathfrak{sl})$ -torsor, $P_{\mathrm{Aut}(\mathbf{par}\mathfrak{sl})}(\mathrm{Out}(\mathbf{par}\mathfrak{sl}))$. The associated bundle is a 2-to-1 cover of U glued to a $(2r)$ -to-1 cover of D through the canonical inclusion $\mathbb{Z}_2 < \mathbb{D}_r$. Since we know that there are reductions of the torsor, the bundle must be the disjoint union of a trivial 2-to-1 cover of X and a trivial $2(r-1)$ cover of D .

We will prove that $\mathrm{Inn}(\mathbf{par}\mathfrak{sl})$ coincides with $\mathcal{G}/\mathbb{C}^* := P\mathcal{G}$. Then, a reduction of $P_{\mathrm{Aut}(\mathbf{par}\mathfrak{sl})}$ to an $\mathrm{Inn}(\mathbf{par}\mathfrak{sl})$ is a parabolic projective bundle $(\mathbb{P}, \mathbb{P}_\bullet) = (\mathbb{P}(E), \mathbb{P}(E_\bullet))$ together with an isomorphism

$$P_{\mathrm{Aut}(\mathbf{par}\mathfrak{sl})} \cong \mathrm{PEnd}_0(\mathbb{P}, \mathbb{P}_\bullet)$$

Let $(\mathbb{P}, \mathbb{P}_\bullet) \rightarrow X$ be a reduction of $P_{\mathrm{Aut}(\mathbf{par}\mathfrak{sl})}$ to $\mathrm{Inn}(\mathbf{par}\mathfrak{sl})$. Then the generator of the \mathbb{Z}_2 component of $\mathrm{Out}(\mathbf{par}\mathfrak{sl})$ corresponds to its dual parabolic projective bundle

$$(\mathbb{P}, \mathbb{P}_\bullet)^\vee = \mathbb{P}((E, E_\bullet)^\vee)$$

On the other hand, by Corollary 4.5.13, for each $x \in D$, the generator of the $\mathbb{Z}_r < \mathbb{D}_r$ outer automorphism corresponds to the Hecke transformation of $(\mathbb{P}, \mathbb{P}_\bullet)$ at the parabolic point $x \in D$. As these outer automorphisms generate $\mathrm{Out}(\mathbf{par}\mathfrak{sl})$, every reduction can be found as a composition of Hecke transformations and dualization of $(\mathbb{P}, \mathbb{P}_\bullet)$.

Now consider the exact sequence of groups

$$1 \longrightarrow Z \longrightarrow \mathcal{G} \longrightarrow \text{Inn}(\mathbf{par}\mathbf{sl}) \longrightarrow 1$$

Let us compute the group scheme Z . As before, over $x \in U$, $\text{Inn}(\mathbf{par}\mathbf{sl})_x = \text{PGL}_r$ and $\mathcal{G}_x = \text{GL}_r$, so $Z_x = \mathbb{C}^*$. Therefore, it is only necessary to compute Z_x for $x \in D$. By definition, Z is the kernel of the adjoint representation. Let $X \in \mathcal{G}_x \hookrightarrow \text{Mat}_{n \times n}(\mathcal{O}_{X,x})$ be in the kernel of the representation. Then, for every $Y \in \mathbf{par}\mathbf{sl}_x \hookrightarrow \text{Mat}_{n \times n}(\mathcal{O}_{X,x})$

$$XY - YX = 0$$

In particular, as given any $G \in \text{End}_0(\mathcal{O}_{X,x})$, $zG \in \mathbf{par}\mathbf{sl}_x$,

$$0 = X(zG) - (zG)X = z(XG - GX)$$

As $\mathcal{O}_{X,x}$ does not have any zero divisors, $XG - GX = 0$ and, therefore, X belongs to the center of $\text{End}_0(\mathcal{O}_{X,x})$, which consists on $\mathcal{O}_{X,x}$ -multiples of the identity. Clearly, all invertible multiples of the identity belong to \mathcal{G}_x and they are in the kernel of the adjoint, so

$$Z_x = \mathcal{O}_{X,x}^*$$

Therefore, we conclude that $Z = \mathbb{C}^* \times X = \mathcal{O}_X^*$ and, taking the quotient, $\text{Inn}(\mathbf{par}\mathbf{sl}) = \mathcal{G}/\mathbb{C}^* := P\mathcal{G}$. As \mathbb{C}^* belongs to the center of \mathcal{G} , the isomorphism classes of reductions of an $\text{Inn}(\mathbf{par}\mathbf{sl})$ -torsor to a \mathcal{G} -torsor form a torsor for the group $H^1(X, \mathcal{O}_X^*)$.

Let (E, E_\bullet) be the parabolic vector bundle corresponding to a reduction of the $P\mathcal{G}$ -torsor $(\mathbb{P}, \mathbb{P}_\bullet) \rightarrow X$, i.e., $(\mathbb{P}(E), \mathbb{P}(E_\bullet)) \cong (\mathbb{P}, \mathbb{P}_\bullet)$. Then the other reductions correspond to parabolic vector bundles of the form $(E, E_\bullet) \otimes L$ for any line bundle L . Similarly, $(E, E_\bullet)^\vee \otimes L$ and $\mathcal{H}_x(E, E_\bullet) \otimes L$ are all the possible reductions of $(\mathbb{P}, \mathbb{P}_\bullet)^\vee$ and $\mathcal{H}_x(\mathbb{P}, \mathbb{P}_\bullet)$ respectively, so all possible reductions can be computed from (E, E_\bullet) by a repeated combination of dualization, tensoring with a line bundle and application of Hecke transformations at parabolic points. \square

4.6 Isomorphisms between moduli spaces of parabolic vector bundles

Let $\Phi : \mathcal{M}(X, r, \alpha, \xi) \rightarrow \mathcal{M}(X', r', \alpha', \xi')$ be an isomorphism between the moduli space of parabolic vector bundles of rank r , determinant ξ and weight system α over (X, D) and the moduli space of parabolic vector bundles of rank r' , determinant ξ' and weight system α' over (X', D') .

By Torelli Theorem 4.3.6, we know that $r = r'$ and that Φ induces an isomorphism between the marked curves $\sigma : (X, D) \xrightarrow{\sim} (X', D')$. We know that the map of quasi-parabolic vector bundles $(E, E_\bullet) \mapsto \sigma^*(E, E_\bullet)$ induces an isomorphism $\Sigma_\sigma : \mathcal{M}(X', r, \alpha', \xi') \rightarrow \mathcal{M}(X, r, \sigma^*\alpha', \sigma^*\xi')$. Therefore, $\Sigma_\sigma \circ \Phi : \mathcal{M}(r, \alpha, \xi) \rightarrow \mathcal{M}(r, \sigma^*\alpha', \sigma^*\xi')$ is an isomorphism between moduli spaces of parabolic vector bundles on (X, D) such that the induced automorphism on the marked curve is the identity. As we can do this for every automorphism of the marked curve, we can assume without loss of generality that Φ induced the identity map on (X, D) .

For $k > 1$, let $W_k = H^0(K^k D^{k-1})$. Recall that we defined

$$h_k : H^0(\text{SPEnd}_0(E) \otimes K_X(D)) \rightarrow W_k$$

as the composition of the Hitchin map $h : H^0(\mathrm{SPEnd}_0(E) \otimes K_X(D)) \rightarrow W$ with the projection $W \twoheadrightarrow W_k$. As we have assumed that Φ induces the identity map on (X, D) , then the Hitchin space for both moduli spaces is the same and by Proposition 4.3.4, there exist a \mathbb{C}^* -equivariant automorphism $f : W \xrightarrow{\sim} W$ such that the following diagram commutes

$$\begin{array}{ccc} T^*\mathcal{M}(r, \alpha, \xi) & \xrightarrow{d(\Phi^{-1})} & T^*\mathcal{M}(r, \alpha', \xi') \\ h \downarrow & & \downarrow h \\ W & \xrightarrow{f} & W \end{array} \quad (4.6.1)$$

Moreover we know that f preserves the block $W_r \subset W$. Our next goal will be to prove that, in fact, there exists linear maps $f_k : W_k \rightarrow W_k$ such that the following diagram commutes for every $k > 1$ (Corollary 4.6.12)

$$\begin{array}{ccc} T^*\mathcal{M}(r, \alpha, \xi) & \xrightarrow{d(\Phi^{-1})} & T^*\mathcal{M}(r, \alpha', \xi') \\ h_r \downarrow & & \downarrow h_r \\ W_k & \xrightarrow{f_k} & W_k \end{array} \quad (4.6.2)$$

in other words, we will prove that $f : W \rightarrow W$ is linear and preserves the decomposition $W = \bigoplus_{k=2}^r W_k$. In order to do so, we will analyze how the geometry of the discriminant $\mathcal{D} \subset W$ and the \mathbb{C}^* -action impose restrictions on the structure of the map $f : W \rightarrow W$.

For every $k > 1$, let us denote

$$W_{\leq k} = \bigoplus_{j=2}^k W_j$$

In particular $W = W_{\leq r}$ and, in order to simplify the notation, we consider $W_{\leq 1} = 0$.

Lemma 4.6.1. *Let $f : W \rightarrow W$ be a \mathbb{C}^* -equivariant isomorphism. If $r = 2$ then f is linear isomorphism. Otherwise, if $r \geq 2$, then there exist*

- An algebraic isomorphism $\bar{g} : W_{\leq(r-2)} \rightarrow W_{\leq(r-2)}$,
- linear isomorphisms $A_j : W_j \rightarrow W_j$, $j = r-1, r$ and
- algebraic maps $g_j : W_{\leq(r-2)} \rightarrow W_j$, $j = r-1, r$

such that for every $s = (\bar{s}, s_{r-1}, s_r) \in W = W_{\leq(r-2)} \oplus W_{r-1} \oplus W_r$

$$f(s_2, \dots, s_r) = (g(\bar{s}), A_{r-1}(s_{r-1}) + g_{r-1}(\bar{s}), A_r(s_r) + g_r(\bar{s}))$$

Proof. Assume that $r \geq 3$ and let $f = (f_2, \dots, f_r)$. Let us fix coordinates $\bar{x}_j = (x_{j,1}, \dots, x_{j,d_j})$ in W_j for each $j = 2, \dots, r$, where

$$d_j = \dim(W_j) = h^0(K^j D^{j-1}) = j(2g-2) + (j-1)n - g + 1$$

In these coordinates, each component of the map $f_j : W \rightarrow W_j$ is written as a weighted-homogeneous polynomial for the weights induced by the \mathbb{C}^* -action. This means that it must be the sum of monomials of the form $C \prod_{i=1}^n x_{j_i, k_i}^{t_i}$ where

$$\sum_{i=1}^l t_i j_i = j \quad t_i > 0, \quad 2 \leq j_i \leq r$$

In particular, the previous equation implies that for every j and every $i = 1, \dots, n$, $j_i \leq j$ and, therefore, the map $f_j : W \rightarrow W_j$ can only depend on variables coming from W_l for $l \leq j$. Moreover, if $j_i = j$ for some i , then there cannot be any other factor in the monomial, i.e., it is a linear monomial. Therefore, each $f_j : W \rightarrow W_j$ decomposes as a sum

$$f_j(s_2, \dots, s_r) = g_j(s_2, \dots, s_{j-1}) + A_j(s_j)$$

for some \mathbb{C}^* -equivariant map $g_j : \bigoplus_{i=2}^{j-1} W_i \rightarrow W_j$ and some linear map $A_j : W_j \rightarrow W_j$. In the particular case $j = r$ we observe that the monomials composing f_r cannot contain the variables $\{x_{j-1, i}\}_{i=1}^{d_{j-1}}$ either, because they have order $r - 1$ for the \mathbb{C}^* action and there does not exist any variable of order 1. Then

$$f_r(s_2, \dots, s_r) = g_r(s_2, \dots, s_{r-2}) + A_r(s_r)$$

Finally, as the inverse f^{-1} must have an analogous decomposition, we conclude that the maps $A_j : W_j \rightarrow W_j$ and the maps

$$\bar{g}_j = (A_2, g_3 + A_3, \dots, g_j + A_j) : \bigoplus_{i=2}^j W_i \longrightarrow \bigoplus_{i=2}^j W_i$$

must be all invertible. The case $r = 2$ is proved in a completely analogous way. \square

Let $\text{sing} : \mathcal{D} \dashrightarrow X$. For each $x \in X$, let $\mathcal{D}_x \subset \mathcal{D}$ be closure of the subset of singular curves which are singular over the point $x \in X$. By definition of the map sing

$$\mathcal{D}_x = \overline{\text{sing}^{-1}(x)}$$

Lemma 4.6.2. *For every $x \in X$, \mathcal{D}_x is a connected rational variety.*

Proof. Let us consider the image of \mathcal{D}_x under the evaluation map

$$0 \longrightarrow \bigoplus_{j=2}^r H^0(K^j D^{j-1}(-2x)) \longrightarrow W \longrightarrow \bigoplus_{j=2}^r K^j D^{j-1} \otimes \mathcal{I}_x / \mathcal{I}_x^2 \longrightarrow 0$$

Then $s \in \mathcal{D}_x$ is the preimage of

$$D = \left\{ (s, s') \in \mathbb{C}^{2(r-1)} \left| \begin{array}{l} \exists t \in \mathbb{C} \quad \begin{array}{l} t^r + \sum_{j=2}^r s_j t^{r-j} = 0 \\ r t^{r-1} + \sum_{j=2}^{r-1} (r-j) s_j t^{r-j-1} = 0 \\ \sum_{j=2}^r s'_j t^{r-j} = 0 \end{array} \end{array} \right. \right\}$$

Clearly, if we prove that D is rational connected, then \mathcal{D}_x is rational connected, as it would be a vector bundle over D . Let us consider the following diagram

$$\begin{array}{ccc}
 \mathbb{C}_t \times \mathbb{C}_s^{r-1} \times \mathbb{C}_{s'}^{r-1} & \xrightarrow{\quad} & \mathbb{C}_t \times \mathbb{C}_s^{r-1} \\
 \swarrow & \downarrow & \swarrow \\
 \tilde{D} & \xrightarrow{\quad} & \tilde{D}_s \\
 \downarrow & \downarrow & \downarrow \\
 \mathbb{C}_s^{r-1} \times \mathbb{C}_{s'}^{r-1} & \xrightarrow{\quad} & \mathbb{C}_s^{r-1} \\
 \swarrow & \downarrow & \swarrow \\
 D & \xrightarrow{\quad} & D_s
 \end{array}$$

where

$$\begin{aligned}
 \tilde{D} &= \left\{ (t, s, s') \in \mathbb{C}^{2r-1} \left| \begin{array}{l} t^r + \sum_{j=2}^r s_j t^{r-j} = 0 \\ rt^{r-1} + \sum_{j=2}^{r-1} (r-j) s_j t^{r-j-1} = 0 \\ \sum_{j=2}^r s'_j t^{r-j} = 0 \end{array} \right. \right\} \\
 \tilde{D}_s &= \left\{ (t, s) \in \mathbb{C}^r \left| \begin{array}{l} t^r + \sum_{j=2}^r s_j t^{r-j} = 0 \\ rt^{r-1} + \sum_{j=2}^{r-1} (r-j) s_j t^{r-j-1} = 0 \end{array} \right. \right\} \\
 D_s &= \left\{ s \in \mathbb{C}^{r-1} \left| \exists t \in \mathbb{C} \begin{array}{l} t^r + \sum_{j=2}^r s_j t^{r-j} = 0 \\ rt^{r-1} + \sum_{j=2}^{r-1} (r-j) s_j t^{r-j-1} = 0 \end{array} \right. \right\}
 \end{aligned}$$

and, clearly, all horizontal and vertical arrows are surjective. Let us consider the open subset $U_s \subset D_s$ corresponding to polynomials of the form $p(x) = (x - t_1)^2(x - t_2) \cdots (x - t_{r-1})$ with $2t_1 + \sum_{i=2}^{r-1} t_i = 0$, all t_i different and $t_1 \neq 0$. Let $U \subset D$, $\tilde{U} \subset \tilde{D}$ and $\tilde{U}_s \subset \tilde{D}_s$ be the preimages of U under the corresponding projection maps. By definition of D_s , it is straightforward to compute that a vector $(\tau, \sigma) \in T_{(t,s)} \mathbb{C}_t \times \mathbb{C}_s^{r-1}$ lines in $T_{(t,s)} \tilde{D}_s$ if and only if

$$\begin{cases} t^r + \sum_{j=2}^r \sigma_j t^{r-j} = 0 \\ rt^{r-1} + \sum_{j=2}^{r-1} (r-j) \sigma_j t^{r-j-1} + \left(r(r-1)t^{r-2} + \sum_{j=2}^{r-2} (r-j)(r-j-1) s_j a^{r-j-2} \right) \tau = 0 \end{cases}$$

Therefore, the differential of the map $\tilde{D}_s \rightarrow D_s$ fails to be injective at (t, s) if and only if

$$r(r-1)t^{r-2} + \sum_{j=2}^{r-2} (r-j)(r-j-1) s_j a^{r-j-2} = 0$$

i.e., if the polynomial $p_s(x) = x^r + \sum_{j=2}^r s_j x^{r-j}$ is divisible by $(x-t)^3$. In particular, if $s \in U_s$, the differential of the map $\tilde{U}_s \rightarrow U_s$ is injective. Moreover, $\tilde{D}_s \rightarrow D_s$ is clearly finite, so the map $\tilde{U}_s \rightarrow U_s$ is a finite and bijective with injective differential. By [Har13, Theorem 14.9 and Corollary 14.10], it is an isomorphism. As points $(t, s) \in \tilde{U}_s$ all have $t \neq 0$, the fiber of the projection $\tilde{U} \rightarrow \tilde{U}_s$ is a vector space of dimension $r-2$ and it is straightforward to check that \tilde{U} is a vector bundle over \tilde{U}_s . Similarly, U is a vector bundle over U_s and it is isomorphic to \tilde{U} through the

isomorphism $\tilde{U}_s \cong U_s$. This proves that D is birational to a vector bundle over \tilde{U}_s . The latter is isomorphic to $\mathbb{C}^* \times \mathbb{C}^{r-3}$ in the following way. Consider \mathbb{C}^k as the space of traceless polynomials $q(x)$ of degree $k+1$. Then \tilde{U}_s is the image of the map

$$\begin{aligned} \mathbb{C}^* \times \mathbb{C}^{r-3} &\longrightarrow \tilde{U}_s \\ (t, q(x)) &\longmapsto (t, (x-t)^2(q(x) + 2tx^{r-3})) \end{aligned}$$

The inverse can be computed through Ruffini's rule, thus inducing an algebraic isomorphism. Therefore D is birational to a vector bundle over $\mathbb{C}^* \times \mathbb{C}^{r-3}$, so it is a connected rational variety. \square

Lemma 4.6.3. *The map $\text{sing} : \mathcal{D} \dashrightarrow X$ commutes with $f : \mathcal{D} \rightarrow \mathcal{D}$.*

Proof. We will proceed as in [BGM12, Remark 4.5]. As $\text{sing} : \mathcal{D} \dashrightarrow X$ has connected rational fibers, there exists a unique such map up to an automorphism of X . Let $\rho : X \rightarrow X$ be the only map such that $f(\mathcal{D}_x) = \mathcal{D}_{\rho(x)}$ for all $x \in X$. The map $f : W \rightarrow W$ preserves W_r and \mathcal{D} , so it preserves $\mathcal{D} \cap W_r = \mathcal{C}_X \cup \bigcup_{x \in D} \mathcal{C}_x$. Moreover, we know that $\mathbb{P}(\mathcal{C}_X)$ is not isomorphic to $\mathbb{P}(\mathcal{C}_x)$ for any $x \in D$, so f must induce an automorphism of \mathcal{C}_X . By construction we assumed that the induced automorphism $\sigma : X \rightarrow X$ on the dual variety is the identity, and it clearly coincides with $\rho : X \rightarrow X$, as for each $x_0 \in X \setminus D$ we have

$$\begin{aligned} H^0(K^r D^{r-1}(-2x_0)) &= f(H^0(K^r D^{r-1}(-2x_0))) = f(\mathcal{D}_x \cap W_r) \\ &= \mathcal{D}_{\rho(x)} \cap W_r = H^0(K^r D^{r-1}(-2\rho(x_0))) \end{aligned}$$

\square

As a consequence, for each $x \in X$, $f(\mathcal{D}_x) = \mathcal{D}_x$. Then, in particular, their intersection is preserved by f . Let

$$\mathcal{N} = \bigcap_{x \in X} \mathcal{D}_x$$

be the subset of spectral curves which are singular over each $x \in X$. The only way this can happen is if the spectral curve is non-reduced, so \mathcal{N} is precisely the set of non-reduced spectral curves. Clearly, it decomposes in irreducible components depending on the degree of the non-reduced factor.

$$\mathcal{N} = \bigcup_{d=1}^{\lfloor r/2 \rfloor} \mathcal{N}^d$$

where

$$\mathcal{N}^d = \left\{ (x^d + a_1 x^{d-1} + \dots + a_d)^2 (x^{r-2d} - 2a_1 x^{r-2d-1} + b_2 x^{r-2d-2} + \dots + b_{r-2d}) \right\}$$

for $d < r/2$ and, if r is even,

$$\mathcal{N}^{r/2} = \left\{ (x^{r/2} + a_2 x^{r/2-2} + \dots + a_{r/2})^2 \right\}$$

with $a_j, b_j \in H^0(K^j D^{j-1})$.

Lemma 4.6.4. *Suppose that $r \geq 3$ and let $f : W \rightarrow W$ be a map such that $f(\mathcal{N}) = \mathcal{N}$. Then it preserves $\mathcal{N}^1 \subset \mathcal{N}$.*

Proof. We will show that the irreducible component $\mathcal{N}^1 \subset \mathcal{N}$ can be identified as the unique irreducible component with the highest dimension. Generically the polynomials $p(x) \in \mathcal{N}^d$ admit a single decomposition as a product $p(x) = p_1(x)^2 p_2(x)$ as above. Therefore, using Riemann-Roch theorem, the dimension of \mathcal{N}^d equals

$$\begin{aligned} \dim(\mathcal{N}^d) &= \sum_{j=1}^d h^0(K^j D^{j-1}) + \sum_{j=2}^{r-2d} h^0(K^j D^{j-1}) = \\ &= \sum_{j=1}^d (j(2g-2) + (j-1)n - g + 1) + \sum_{j=2}^{r-2d} (j(2g-2) + (j-1)n - g + 1) = \\ &= d^2(g-1) + n \frac{d(d-1)}{2} + ((r-2d)^2 - 1)(g-1) + n \frac{(r-2d)(r-2d-1)}{2} \end{aligned}$$

for $d < r/2$ and, if $r > 2$ is even, then

$$\dim(\mathcal{N}^{r/2}) = \sum_{j=2}^{r/2} h^0(K^j D^{j-1}) = ((r/2)^2 - 1)(g-1) + n \frac{(r/2)(r/2-1)}{2}$$

Observe that for every d with $1 < d < r/2$

$$\begin{aligned} \dim(\mathcal{N}^d) &= \sum_{j=1}^d h^0(K^j D^{j-1}) + \sum_{j=2}^{r-2d} h^0(K^j D^{j-1}) \leq \\ &= h^0(K) + \sum_{j=r-d}^{r-2} h^0(K^j D^{j-1}) + \sum_{j=2}^{r-2d} h^0(K^j D^{j-1}) < \sum_{j=1}^{r-2} h^0(K^j D^{j-1}) = \dim(\mathcal{N}^1) \end{aligned}$$

and, if $r > 2$ is even then $r/2 \leq r-2$, so clearly

$$\dim(\mathcal{N}^{r/2}) = \sum_{j=2}^{r/2} h^0(K^j D^{j-1}) \leq \sum_{j=2}^{r-2} h^0(K^j D^{j-1}) < \sum_{j=1}^{r-2} h^0(K^j D^{j-1}) = \dim(\mathcal{N}^1)$$

so \mathcal{N}^1 is the irreducible component of \mathcal{N} of maximum dimension, and it is the only component with such dimension. Therefore, $f(\mathcal{N}^1) = \mathcal{N}^1$. \square

For each $a \in H^0(K)$, let

$$\mathcal{N}^1(a) = \{(x-a)^2(x^{r-2} + 2ax^{r-3} + b_2x^{r-4} + \dots + b_{r-2})\}$$

where $b_j \in H^0(K^j D^{j-1})$. Then, by definition

$$\mathcal{N}^1 = \bigcup_{a \in H^0(K)} \mathcal{N}^1(a)$$

We can analyze the geometry of \mathcal{N}^1 in terms of $\mathcal{N}^1(a)$. Let us consider the following intersection variety constructed as a disjoint union of the slices $\mathcal{N}^1(a)$

$$\mathcal{I} = \coprod_{a \in H^0(K)} \{a\} \times \mathcal{N}^1(a) \subsetneq H^0(K) \times W$$

It is clearly the preimage of 0 under the map

$$\begin{aligned} F : H^0(K) \times W &\longrightarrow H^0(K^{r-1}D^{r-2}) \times H^0(K^rD^{r-1}) \\ (a, s_2, \dots, s_r) &\longmapsto \begin{pmatrix} ra^{r-1} + \sum_{k=2}^{r-1} (r-k)s_k a^{r-k-1} \\ a^r + \sum_{k=2}^r s_k a^{r-k} \end{pmatrix} \end{aligned}$$

Lemma 4.6.5. *There exist a basis $\{w_i\}$ of $H^0(K)$ such that the sections $w_i^r \in H^0(K^rD^{r-1})$ are linearly independent.*

Proof. Assume that the lemma is false. Then let us prove that the image of $H^0(K)$ under the algebraic map

$$H^0(K) \xrightarrow{(\cdot)^r} H^0(K^rD^{r-1})$$

is contained in some linear subspace $V \subset H^0(K^rD^{r-1})$ of dimension at most $g-1$. Let $m < g$ be the maximum rank of the images of a basis $\{w_1, \dots, w_g\} \subset H^0(K)$. Then there is some basis $\{w_1, \dots, w_g\}$ such that for each $i > m$, w_i^r belongs to the m -dimensional linear space

$$V = \text{Span}(\{w_j^r\}_{j \leq m}) \subset H^0(K^rD^{r-1})$$

In particular, as $\{w_j^r\}_{j \leq m}$ generate a subspace of the maximum dimension, the images of the vectors of any other basis containing $\{w_j\}_{j \leq m}$, must be contained in V . In particular, if we pick any $w'_g \in U = H^0(K) \setminus \text{Span}(\{w_j\}_{j < g})$, then $\{w_1, \dots, w_{g-1}, w'_g\}$ is a basis of $H^0(K)$ and we get that $(w'_g)^r \in V$. Therefore, the image of the open subset $U = H^0(K) \setminus \text{Span}(\{w_j\}_{j < g}) \subset H^0(K)$ is contained in V . As U is dense and the map $H^0(K) \rightarrow H^0(K^rD^{r-1})$ is continuous, the whole image of the map must be contained in V . Then, by upper semicontinuity of the dimension of the fibers, all the fibers of the algebraic map $H^0(K) \rightarrow V$ must have dimension at least 1. In particular, there must exist a nonzero $w \in H^0(K)$ such that $w^r = 0$, but this is impossible. \square

Let $\pi_1 : \mathcal{I} \rightarrow H^0(K)$ and $\pi_2 : \mathcal{I} \rightarrow \mathcal{N}^1$ be the canonical projections.

Lemma 4.6.6. *The map $\pi_2 : \mathcal{I} \rightarrow \mathcal{N}^1$ sending $(a, s) \mapsto s$ is a finite map.*

Proof. The fibers of the map are clearly finite, so it is only necessary to prove that $\mathbb{C}[\mathcal{I}]$ is a finite algebra over $\mathbb{C}[\mathcal{N}^1]$. We know that $\mathcal{I} \subset H^0(K) \times W$ is defined by the equations $F(a, \bar{s}) = 0$. Let $I_{\mathcal{N}^1}$ be the ideal defining $\mathcal{N}^1 \subset W$ and let $\{w_i\}_{i=1}^g$ be a basis of $H^0(K)$ as in the Lemma 4.6.5. Then it is straightforward to check that

$$\mathbb{C}[\mathcal{I}] \cong \frac{\mathbb{C}[W][t_1, \dots, t_g]}{I_{\mathcal{N}^1} + I}$$

where $I \subset \mathbb{C}[W][t_1, \dots, t_g]$ is the ideal generated by each of the components of the vector

$$\left(\left(\sum_{i=1}^g t_i w_i \right)^{r-1} - \sum_{k=2}^{r-1} (r-k) s_k \left(\sum_{i=1}^g t_i w_i \right)^{r-k-1}, \left(\sum_{i=1}^g t_i w_i \right)^r - \sum_{k=2}^r s_k \left(\sum_{i=1}^g t_i w_i \right)^{r-k} \right)$$

in any basis of $H^0(K^{r-1}D^{r-2}) \oplus H^0(K^r D^{r-1})$ extending $\{w_i^r\}$.

Therefore, in order to prove that $\mathbb{C}[Z]$ is a finitely generated $\mathbb{C}[\mathcal{N}^1] = \mathbb{C}[W]/I_{\mathcal{N}^1}$ -module, it is enough to find a relation in I between t_i^r and lower order terms t_j^k with $j < r$ and coefficients in $\mathbb{C}[\mathcal{N}^1]$. Observe that

$$\left(\sum_{i=1}^g t_i w_i \right)^r - \sum_{k=2}^r s_k \left(\sum_{i=1}^g t_i w_i \right)^{r-k} = \sum_{i=1}^g t_i^r w_i^r + O(\{t_j^{r-1}\})$$

As w_i^r are linearly independent, taking the w_i^r coordinate of this vector we obtain an expression of the form $t_i^r + O(\{t_j^{r-1}\})$ which, by construction, has coefficients in $\mathbb{C}[\mathcal{N}^1]$ and belongs to I for every $i = 1, \dots, g$. Therefore, $\mathbb{C}[Z]$ is generated as a $\mathbb{C}[\mathcal{N}^1]$ -module by $\{t_1^{j_1} \dots t_g^{j_g} | j_i < r\}$. \square

Lemma 4.6.7. *There is an open nonempty set $\mathcal{U}^{\text{sm}} \subset \mathcal{N}^1$ such that the differential of the map $\pi_2 : \pi_2^{-1}(\mathcal{U}^{\text{sm}}) \rightarrow \mathcal{N}^1$ is invertible at every point.*

Proof. The differential of the map π_2 is invertible over the points $s \in \mathcal{N}^1$ such that $H^0(K)$ is transverse to $T_{(a,s)}\mathcal{I} \subset H^0(K) \oplus W$. Let us compute the tangent space to \mathcal{I} . By construction it is the kernel of the differential of F

$$dF : H^0(K) \oplus W \longrightarrow H^0(K^{r-1}D^{r-2}) \oplus H^0(K^r D^{r-1})$$

It is straightforward to compute the differential at a point (a, s) from the equations of F . If $(\alpha, \sigma_2, \dots, \sigma_r) \in T_{(a,s)}H^0(K) \times W \cong H^0(K) \oplus W$, then

$$dF(\alpha, \sigma_2, \dots, \sigma_r) = \begin{pmatrix} (ra^{r-1} + \sum_{k=2}^{r-1} (r-k)\sigma_k a^{r-k-1}) + (r(r-1)a^{r-2} + \sum_{k=2}^{r-2} (r-k)(r-k-1)s_k a^{r-k-2})\alpha \\ (a^r + \sum_{k=2}^r \sigma_k a^{r-k}) + (ra^{r-1} + \sum_{k=2}^{r-1} (r-k)s_k a^{r-k-1})\alpha \end{pmatrix}$$

As $(a, s) \in \mathcal{I} = F^{-1}(0)$ the last summand in the second component is zero, so the equations of $T_{(a,s)}\mathcal{I}$ become

$$\begin{cases} ra^{r-1} + \sum_{k=2}^{r-1} (r-k)\sigma_k a^{r-k-1} + (r(r-1)a^{r-2} + \sum_{k=2}^{r-2} (r-k)(r-k-1)s_k a^{r-k-2})\alpha = 0 \\ a^r + \sum_{k=2}^r \sigma_k a^{r-k} = 0 \end{cases}$$

Therefore, $H^0(K)$ fails to be transverse to $T_{(a,s)}\mathcal{I}$ if and only if

$$r(r-1)a^{r-2} + \sum_{k=2}^{r-2} (r-k)(r-k-1)s_k a^{r-k-2} = 0$$

This, together with the assumption that $F(a, s) = 0$, implies that the polynomial corresponding to s admits a decomposition

$$p_s(x) = (x-a)^3 q(x)$$

for some q . Repeating the dimension counting argument in Lemma 4.6.4 we obtain that the set of point admitting such decomposition has positive codimension in \mathcal{N}^1 , so its complement $\mathcal{U}^{\text{sm}} \subset \mathcal{N}^1$ is an open nonempty set. For $r = 2$, $\mathcal{U}^{\text{sm}} = \mathcal{N}^1$, for $r = 3$, $\mathcal{U}^{\text{sm}} = \mathcal{N}^1 \setminus \{0\}$ and for $r > 3$

$$\dim(\mathcal{N}^1 \setminus \mathcal{U}^{\text{sm}}) = h^0(K) + \sum_{j=2}^{r-3} h^0(K^j D^{j-1}) < \sum_{j=1}^{r-2} h^0(K^j D^{j-1}) = \dim(\mathcal{N}^1)$$

□

Lemma 4.6.8. *The projection $\pi_2 : \mathcal{I} \rightarrow \mathcal{N}^1$ is a \mathbb{C}^* -equivariant birational map.*

Proof. The space of points in \mathcal{N}^1 admitting at least a decomposition of the form

$$p(x) = (x - a)^2 q(x)$$

for at least two different sections $a \in H^0$ corresponds to the points in \mathcal{N}^1 admitting a decomposition of the form

$$p(x) = (x - a)^2 (x - b)^2 q(x)$$

For some a, b . Again, repeating the dimension argument used in Lemma 4.6.4, we obtain that the dimension of this subset is less than the dimension of \mathcal{N}^1 . Let \mathcal{U}^{bi} denote its complement in \mathcal{N}^1 . Then for $r < 4$, $\mathcal{U}^{\text{bi}} = \mathcal{N}^1$. For $r = 4$

$$\dim(\mathcal{N}^1 \setminus \mathcal{U}^{\text{bi}}) = h^0(K) < h^0(K) + h^0(K^2 D^1) = \dim(\mathcal{N}^1)$$

and for $r > 4$

$$\begin{aligned} \dim(\mathcal{N}^1 \setminus \mathcal{U}^{\text{bi}}) &= 2h^0(K) + \sum_{j=2}^{r-4} h^0(K^j D^{j-1}) \\ &= \sum_{j=1}^{r-3} h^0(K^j D^{j-1}) - (h^0(K^{r-3} D^{r-4}) - h^0(K)) < \sum_{j=1}^{r-2} h^0(K^j D^{j-1}) = \dim(\mathcal{N}^1) \end{aligned}$$

Therefore, there exists an open nonempty subset $\mathcal{U}^{\text{bi}} \subset \mathcal{N}^1$ consisting on points s whose preimage $\pi_2^{-1}(s)$ is a single point. On the other hand, by Lemma 4.6.7, there exist a subset \mathcal{U}^{sm} such that the differential of the map $\pi_2|_{\mathcal{U}^{\text{sm}}}$ is invertible. By Lemma 4.6.6, we know that π_2 is a finite map, so restricting it to $\mathcal{U} = \mathcal{U}^{\text{bi}} \cap \mathcal{U}^{\text{sm}}$, we obtain a finite bijective map with invertible differential. By [Har13, Theorem 14.9 and Corollary 14.10], $\pi_2|_{\pi_2^{-1}(\mathcal{U})} : \pi_2^{-1}(\mathcal{U}) \rightarrow \mathcal{U}$ is an isomorphism and, therefore, it induces a birational map between \mathcal{I} and \mathcal{N}^1 . □

Lemma 4.6.9. *Suppose that $r \geq 3$. Let $f : W \rightarrow W$ be a \mathbb{C}^* -equivariant isomorphism such that $f(\mathcal{N}) = \mathcal{N}$. Then there is a \mathbb{C}^* -equivariant isomorphism $g : W_{\leq(r-2)} \rightarrow W_{\leq(r-2)}$ and linear maps $f_j : W_j \rightarrow W_j$ for $j = r-1, r$ such that the following diagrams commute*

$$\begin{array}{ccccc} W & \xrightarrow{f} & W & & W & \xrightarrow{f} & W & & W & \xrightarrow{f} & W \\ \pi_{\leq(r-2)} \downarrow & & \downarrow \pi_{\leq(r-2)} & & \pi_{r-1} \downarrow & & \downarrow \pi_{r-1} & & \pi_r \downarrow & & \downarrow \pi_r \\ W_{\leq(r-2)} & \xrightarrow{g} & W_{\leq(r-2)} & & W_{r-1} & \xrightarrow{f_{r-1}} & W_{r-1} & & W_r & \xrightarrow{f_r} & W_r \end{array}$$

Proof. Taking into account the block decomposition in Lemma 4.6.1, it is enough to prove that the map $(g_{r-1}, g_r) : W_{\leq(r-2)} \rightarrow W_{r-1} \oplus W_r$ is zero. By definition, $W_{\leq(r-2)} = \mathcal{N}^1(0) \subset \mathcal{N}^1$, so Lemma 4.6.4 implies that $f(W_{\leq(r-2)}) \subset \mathcal{N}^1$. The dimension of $W_{\leq(r-2)}$ is

$$\dim(W_{\leq(r-2)}) = \sum_{j=2}^{r-2} h^0(K^j D^{j-1})$$

so comparing it with the dimensions of $\mathcal{N}^1 \setminus \mathcal{U}^{\text{sm}}$ and $\mathcal{N}^1 \setminus \mathcal{U}^{\text{bi}}$ computed in Lemmas 4.6.7 and 4.6.8, we obtain that $\dim(\mathcal{N}^1 \setminus \mathcal{U}^{\text{sm}}) < \dim(W_{\leq(r-2)})$ and $\dim(\mathcal{N}^1 \setminus \mathcal{U}^{\text{bi}}) < \dim(W_{\leq(r-2)})$. Therefore, $\mathcal{U} \cap f(W_{\leq(r-2)}) = \mathcal{U}^{\text{sm}} \cap \mathcal{U}^{\text{bi}} \cap f(W_{\leq(r-2)})$ is an open dense subset of $f(W_{\leq(r-2)})$.

Thus, applying Lemma 4.6.8, we obtain a \mathbb{C}^* -equivariant rational map $f(W_{\leq(r-2)}) \dashrightarrow \mathcal{I}$. On the other hand, let us consider the isomorphism $\bar{g} : W_{\leq(r-2)} \rightarrow W_{\leq(r-2)}$ given by the decomposition in blocks of $f : W \rightarrow W$ described in Lemma 4.6.1. Composing the rational map $f(W_{\leq(r-2)}) \dashrightarrow \mathcal{I}$ with the canonical projection, $\pi_1 : \mathcal{I} \rightarrow H^0(K)$, and the map $\tilde{f} = f \circ \bar{g}^{-1} : W_{\leq(r-2)} \rightarrow f(W_{\leq(r-2)})$, we obtain a rational map $t : W_{\leq(r-2)} \dashrightarrow H^0(K)$, satisfying the following property. Let $(\bar{s}, s_{r-1}, s_r) = (s_2, \dots, s_r) \in f(W_{\leq(r-2)})$ be a generic point. Then $t(\bar{s}) \in H^0(K)$ is the only section such that

$$\begin{cases} t(\bar{s})^r + \sum_{j=2}^r s_j t(\bar{s})^{r-j} = 0 \\ rt(\bar{s})^{r-1} + \sum_{j=2}^{r-1} (r-j) s_j t(\bar{s})^{r-j} = 0 \end{cases}$$

In particular, solving for s_{r-1} and s_r we obtain that

$$\begin{cases} s_{r-1} = rt(\bar{s})^{r-1} + \sum_{j=2}^{r-2} (r-j) s_j t(\bar{s})^{r-j-1} \\ s_r = (r-1)t(\bar{s})^r + \sum_{j=2}^{r-2} (r-j-1) s_j t(\bar{s})^{r-j} \end{cases}$$

On the other hand, as $(\bar{s}, s_{r-1}, s_r) \in f(W_{\leq(r-2)})$, by the block decomposition we know that

$$\begin{cases} s_{r-1} = g_{r-1} \circ \bar{g}^{-1}(\bar{s}) := \tilde{g}_{r-1}(\bar{s}) \\ s_r = g_r \circ \bar{g}^{-1}(\bar{s}) := \tilde{g}_r(\bar{s}) \end{cases}$$

so

$$\begin{cases} \tilde{g}_{r-1}(\bar{s}) = rt(\bar{s})^{r-1} + \sum_{j=2}^{r-2} (r-j) s_j t(\bar{s})^{r-j-1} \\ \tilde{g}_r(\bar{s}) = (r-1)t(\bar{s})^r + \sum_{j=2}^{r-2} (r-j-1) s_j t(\bar{s})^{r-j} \end{cases} \quad (4.6.3)$$

As $t : W_{\leq(r-2)} \dashrightarrow H^0(K)$ is a \mathbb{C}^* -equivariant rational map between vector spaces there are three possibilities for its structure

1. $t = 0$, in which case we would get $g_{r-1} = 0$ and $g_r = 0$ leading to the desired result.
2. $t : W_{\leq(r-2)} \rightarrow H^0(K)$ is a homogeneous polynomial. This is impossible because the action of \mathbb{C}^* in $W_{\leq(r-2)}$ is of order at least 2 and the action of \mathbb{C}^* in $H^0(K)$ has order 1.
3. $t(\bar{s}) = \frac{\alpha(\bar{s})}{\beta(\bar{s})}$ for some homogeneous polynomials α and β with no common factors.

Then it is only left to prove that (3) is also impossible. Substituting $t = \alpha/\beta$ in (4.6.3) we obtain the following equality

$$\begin{cases} \beta(\bar{s})^{r-1} \tilde{g}_{r-1}(\bar{s}) = r\alpha(\bar{s})^{r-1} + \sum_{j=2}^{r-2} (r-j)s_j \alpha(\bar{s})^{r-j-1} \beta(\bar{s})^j \\ \beta(\bar{s})^r \tilde{g}_r(\bar{s}) = (r-1)\alpha(\bar{s})^r + \sum_{j=2}^{r-2} (r-j-1)s_j \alpha(\bar{s})^{r-j} \beta(\bar{s})^j \end{cases}$$

Nevertheless, looking at the last equation modulo β we get that α^r is a multiple of β , thus contradicting that α and β do not share a common factor. \square

In order to prove that f is linear and decomposes diagonally, we will apply the previous lemma inductively. For each $k > 1$, let $\mathcal{N}_k \subset \bigoplus_{j=2}^k H^0(K^j D^{j-1})$ be the set of non-reduced “rank k ” spectral curves, i.e., the set of spectral curves defined by degree k polynomials of the form

$$x^k + \sum_{j=2}^k s_j x^{k-j} = 0$$

for $s_j \in H^0(K^j D^{j-1})$ which have at least a non-reduced component. With a slight abuse of notation let us also denote by $\mathcal{N}_k \subset W_{\leq k}$ the image of the set of rank k non-reduced spectral curves under the inclusion

$$\mathcal{N}_k \subseteq \bigoplus_{j=2}^k H^0(K^j D^{j-1}) \subseteq \bigoplus_{j=2}^r H^0(K^j D^{j-1})$$

In other words,

$$\mathcal{N}_k = \{x^{r-k}q(x) \mid q(x) = p_1(x)^2 p_2(x) \text{ for some } p_1(x) \text{ and } p_2(x)\}$$

Lemma 4.6.10. *Let $k \geq 3$ and let $f_{\leq k} : W_{\leq k} \rightarrow W_{\leq k}$ be a \mathbb{C}^* -equivariant isomorphism such that $f_{\leq k}(\mathcal{N}_k) = \mathcal{N}_k$. Then there is a \mathbb{C}^* -equivariant isomorphism $f_{\leq(k-2)} : W_{\leq(k-2)} \rightarrow W_{\leq(k-2)}$ and linear maps $f_j : W_j \rightarrow W_j$ for $j = k-1, k$ such that $f_{\leq(k-2)}(\mathcal{N}_{k-2}) = \mathcal{N}_{k-2}$ and the following diagrams commute*

$$\begin{array}{ccccc} W_{\leq k} & \xrightarrow{f_{\leq k}} & W_{\leq k} & & W_{\leq k} & \xrightarrow{f_{\leq k}} & W_{\leq k} \\ \pi_{\leq(k-2)} \downarrow & & \downarrow \pi_{\leq(k-2)} & & \pi_{k-1} \downarrow & & \downarrow \pi_{k-1} \\ W_{\leq(k-2)} & \xrightarrow{f_{\leq(k-2)}} & W_{\leq(k-2)} & & W_{k-1} & \xrightarrow{f_{k-1}} & W_{k-1} \\ & & & & & & \downarrow \pi_{k-1} \\ & & & & W_k & \xrightarrow{f_k} & W_k \end{array}$$

Proof. Applying the Lemma 4.6.9 to $r = k$, we obtain the desired diagonal decomposition $f_{\leq k} = (f_{\leq(k-2)}, f_{k-1}, f_k) : W_{\leq(k-2)} \oplus W_{k-1} \oplus W_k \rightarrow W_{\leq(k-2)} \oplus W_{k-1} \oplus W_k$. Therefore, it is enough to prove that $f_{\leq(k-2)}$ preserves \mathcal{N}_{k-2} . We know that \mathcal{N}_k decomposes in irreducible components as

$$\mathcal{N}_k = \bigcup_{d=1}^{\lfloor k/2 \rfloor} \mathcal{N}_k^d$$

where

$$\mathcal{N}_k^d = \left\{ (x^d + a_1 x^{d-1} + \dots + a_d)^2 (x^{k-2d} - 2a_1 x^{k-2d-1} + b_2 x^{k-2d-2} + \dots + b_{k-2d}) \right\}$$

for $d < k/2$ and, if k is even,

$$\mathcal{N}_k^{k/2} = \left\{ (x^{k/2} + a_2 x^{k/2-2} + \dots + a_{k/2})^2 \right\}$$

By hypothesis we know that $f_{\leq k}(\mathcal{N}_k) = \mathcal{N}_k$ and, by Lemma 4.6.4, $f_{\leq k}(\mathcal{N}_k^1) = \mathcal{N}_k^1$, so, $f_{\leq k}$ must preserve the union of the rest of the components.

$$f_{\leq k} \left(\bigcup_{d=2}^{\lfloor k/2 \rfloor} \mathcal{N}_k^d \right) = \bigcup_{d=2}^{\lfloor k/2 \rfloor} \mathcal{N}_k^d$$

On the other hand, for each $d > 1$ consider the intersection $W_{\leq (k-2)} \cap \mathcal{N}_k^d \subset W_{\leq k}$. The elements in $W_{\leq (k-2)}$ correspond to polynomials $p(x) \in W_{\leq k}$ which have at least a factor x^2 , i.e.

$$W_{\leq (k-2)} = \{x^2 q(x) \in W_{\leq k}\}$$

On the other hand, the elements in \mathcal{N}_k^d are polynomials with at least a double factor of order d

$$\mathcal{N}_k^d = \{p_1(x)^2 p_2(x) \mid \deg(p_1) = d\}$$

Then when we get the intersection, for each polynomial of the form $p(x) = p_1(x)^2 p_2(x) \in W_{\leq (k-2)} \cap \mathcal{N}_k^d$ there are two possibilities

1. Either the x^2 factor is included in $p_1(x)$, so $p_1(x) = x q_1(x)$ for some q of degree $d-1$ and then

$$p(x) = x^2 q_1(x)^2 p_2(x) \in \mathcal{N}_{k-2}^{d-1}$$

2. or the x^2 factor is included in $p_2(x)$, so $p_2(x) = x^2 q_2(x)$ and then

$$p(x) = x^2 p_1(x)^2 q_2(x) \in \mathcal{N}_{k-2}^d$$

and the latter can only happen if $d \leq (k-2)/2$. Therefore, we conclude that

$$W_{\leq (k-2)} \cap \mathcal{N}_k^d = \begin{cases} \mathcal{N}_{k-2}^{d-1} \cup \mathcal{N}_{k-2}^d & d \leq (k-2)/2 \\ \mathcal{N}_{k-2}^{d-1} & d > (k-2)/2 \end{cases}$$

In particular, taking the full union for $d > 1$ yields

$$W_{\leq (k-2)} \cap \bigcup_{d=2}^{\lfloor k/2 \rfloor} \mathcal{N}_k^d = \bigcup_{d=1}^{\lfloor (k-2)/2 \rfloor} \mathcal{N}_{k-2}^d = \mathcal{N}_{k-2}$$

As $f_{\leq k}$ preserves both $W_{\leq (k-2)}$ and the union of the components \mathcal{N}_k^d for $d > 1$, we obtain that $f_{\leq k}(\mathcal{N}_{k-2}) = \mathcal{N}_{k-2}$. Finally, as $\mathcal{N}_{k-2} \subset W_{\leq (k-2)}$ and we already know that f decomposes diagonally with respect to the last two factors W_{k-1} and W_k , then $f_{\leq (k-2)}(\mathcal{N}_{k-2}) = \mathcal{N}_{k-2}$. \square

Now we can apply the previous lemma inductively and combine it with the previous results to recover the diagonal decomposition.

Lemma 4.6.11. *Let $f : W \rightarrow W$ be a \mathbb{C}^* -equivariant isomorphism such that $f(\mathcal{D}) = \mathcal{D}$. Then for every $k > 1$, there exist a linear automorphism $f_k : W_k \rightarrow W_k$ such that the following diagram commutes*

$$\begin{array}{ccc} W & \xrightarrow{f} & W \\ \pi_k \downarrow & & \downarrow \pi_k \\ W_k & \xrightarrow{f_k} & W_k \end{array} \quad (4.6.4)$$

Proof. By Lemma 4.6.3, the map $\text{sing} : \mathcal{D} \dashrightarrow X$ commutes with $f : \mathcal{D} \rightarrow \mathcal{D}$, so f preserves the closure of the fibers $\mathcal{D}_x = \overline{\text{sing}^{-1}(x)}$. Then, it preserves its intersection, but we know by construction that $\mathcal{N}_r = \bigcap_{x \in X} \mathcal{D}_x$, so $f(\mathcal{N}_r) = \mathcal{N}_r$. Moreover, $f : W \rightarrow W$ is \mathbb{C}^* -equivariant by hypothesis, so we can apply Lemma 4.6.10 and we obtain that $f = f_{\leq r}$ commutes with the projections into $W_{\leq k-2}$, W_{r-1} and W_r , decomposing diagonally as

$$f_{\leq r} = (f_{\leq(r-2)}, f_{r-1}, f_r) : W_{\leq(r-2)} \oplus W_{r-1} \oplus W_r \longrightarrow W_{\leq(r-2)} \oplus W_{r-1} \oplus W_r$$

with f_{r-1} and f_r linear maps. Moreover $f_{\leq(r-2)}(\mathcal{N}_{r-2}) = \mathcal{N}_{r-2}$. Now we can restrict ourselves to $W_{\leq(r-2)}$. We proved that we have a \mathbb{C}^* -equivariant isomorphism $f_{\leq(r-2)} : W_{\leq(r-2)} \rightarrow W_{\leq(r-2)}$ such that $f_{\leq(r-2)}(\mathcal{N}_{r-2}) = \mathcal{N}_{r-2}$, so we can apply Lemma 4.6.10 again and find that $f_{\leq(r-2)}$ decomposes as

$$f_{\leq(r-2)} = (f_{\leq(r-4)}, f_{r-3}, f_{r-2}) : W_{\leq(r-4)} \oplus W_{r-3} \oplus W_{r-2} \longrightarrow W_{\leq(r-4)} \oplus W_{r-3} \oplus W_{r-2}$$

and, moreover $f_{\leq(r-4)}(\mathcal{N}_{r-4}) = \mathcal{N}_{r-4}$. This together with the previous part proves that $f : W \rightarrow W$ decomposes as

$$f = (f_{\leq(r-4)}, f_{r-3}, \dots, f_r) : W_{\leq(r-4)} \oplus W_{r-3} \oplus \dots \oplus W_r \longrightarrow W_{\leq(r-4)} \oplus W_{r-3} \oplus \dots \oplus W_r$$

Where f_j are linear for $j \geq r-3$. Repeating this argument successively, we arrive to two different situations depending on the parity of r .

If r is even, we arrive to a diagonal decomposition $f = (f_2, \dots, f_r)$ with $f_j : W_j \rightarrow W_j$ linear, so we are done. If r is odd, we obtain a diagonal decomposition $f = (f_{\leq 2}, f_3, \dots, f_r)$ with $f_j : W_j \rightarrow W_j$ linear for each $j > 2$ and $f_{\leq 2} : W_2 \rightarrow W_2$ a \mathbb{C}^* -equivariant isomorphism. Then, simply apply the $r = 2$ case of Lemma 4.6.1 to $f_{\leq 2}$ to prove that it is a linear isomorphism. \square

In particular, combining the previous lemma with diagram 4.6.1, we obtain

Corollary 4.6.12. *For every $k > 1$, there exist a linear automorphism $f_k : W_k \rightarrow W_k$ such that the following diagram commutes*

$$\begin{array}{ccc} T^*\mathcal{M}(r, \alpha, \xi) & \xrightarrow{d(\Phi^{-1})} & T^*\mathcal{M}(r, \alpha', \xi') \\ h_k \downarrow & & \downarrow h_k \\ W_k & \xrightarrow{f_k} & W_k \end{array} \quad (4.6.5)$$

Once we have characterized $f_r : W_r \rightarrow W_r$, we can further state the following Lemma.

Lemma 4.6.13. *Let $f_r : W_r \rightarrow W_r$ be the linear automorphism constructed in Corollary 4.6.12. Then for every $k > 0$ and every $x_0 \in X$*

$$f_r(H^0(K^r D^{r-1}(-kx_0))) = H^0(K^r D^{r-1}(-kx_0))$$

Proof. As $d(\Phi^{-1})$ is an isomorphism, it maps complete rational curves on the cotangent bundle to complete rational curves. By Lemma 4.3.3, the morphism f must preserve $\mathcal{C} = \mathcal{D} \cap W_r$. Applying 4.3.5, we can decompose $\mathcal{C} = \mathcal{C}_X \cup \bigcup_{x \in D} \mathcal{C}_x$, where $\mathbb{P}(\mathcal{C}_X)$ is the dual variety of $X \subset \mathbb{P}(W_r^*)$ and $\mathbb{P}(\mathcal{C}_x)$ is the set of hyperplanes going through $x \in X \subset \mathbb{P}(W_r^*)$. Moreover, we know that $\mathbb{P}(\mathcal{C}_X)$ is not isomorphic to $\mathbb{P}(\mathcal{C}_x)$ for any x , so f must preserve \mathcal{C}_X . As we assumed that the induced automorphism of the dual variety $\sigma : X \rightarrow X$ is the identity, then for each $x_0 \in X$, the projectivization of f must preserve all the osculating spaces at x_0 . The osculating k space at $x_0 \in X \subset \mathbb{P}(W_r^*)$ is precisely $\mathbb{P}(H^0(K^r D^{r-1}(-kx_0)))$. As $f : W_r \rightarrow W_r$ is linear, we conclude that it preserves $H^0(K^r D^{r-1}(-kx_0))$. \square

Lemma 4.6.14. *Suppose that $g \geq 4$. Let $(E, E_\bullet) \in \mathcal{M}(r, \alpha, \xi)$ and $(E', E'_\bullet) \in \mathcal{M}(r, \alpha', \xi')$ be generic stable parabolic vector bundles such that $\Phi(E, E_\bullet) = (E', E'_\bullet)$. Consider the isomorphism of vector spaces*

$$d(\Phi^{-1}) : H^0(\mathrm{SPEnd}_0(E) \otimes K_X(D)) \longrightarrow H^0(\mathrm{SPEnd}_0(E') \otimes K_X(D))$$

Then for every $x \in U$, the image of $H^0(\mathrm{SPEnd}_0(E) \otimes K_X(D - x))$ under $d(\Phi^{-1})$ is $H^0(\mathrm{SPEnd}_0(E') \otimes K_X(D - x))$.

Proof. Let $x_0 \in U$. Let (E, E_\bullet) be a generic stable parabolic vector bundle in the sense of Lemma 4.1.7. We will prove that

$$H^0(\mathrm{SPEnd}_0(E) \otimes K_X(D - x)) = \{\psi \in H_{x_0} : \forall \varphi \in h_r^{-1}(H_{x_0}) \ h_r(\psi + \varphi) \in H_{x_0}\}$$

where $H_{x_0} = H^0(K^r D^{r-1}(-x_0)) \subseteq W_r = H^0(K^r D^{r-1})$. By Lemma 4.6.13, H_{x_0} is preserved by $f_r : W_r \rightarrow W_r$, so the Lemma follows from commutativity of diagram (4.6.2).

As we assumed $g \geq 4$, by Lemma 4.1.7, for a generic (E, E_\bullet)

$$H^1(\mathrm{SPEnd}_0(E, E_\bullet) \otimes K(D - x_0)) = H^0(\mathrm{PEnd}_0(E, E_\bullet)(x_0))^\vee = 0$$

Therefore, for a generic parabolic vector bundle the following sequence is exact

$$\begin{aligned} 0 \longrightarrow H^0(\mathrm{SPEnd}_0(E, E_\bullet) \otimes K(D - x_0)) &\longrightarrow H^0(\mathrm{SPEnd}_0(E, E_\bullet) \otimes K(D)) \\ &\longrightarrow \mathrm{SPEnd}_0(E, E_\bullet) \otimes K(D)|_{x_0} \longrightarrow 0 \end{aligned}$$

Therefore, the evaluation map

$$H^0(\mathrm{SPEnd}_0(E, E_\bullet) \otimes K(D)) \twoheadrightarrow \mathrm{SPEnd}_0(E, E_\bullet) \otimes K(D)|_{x_0} \cong \mathrm{End}(E)|_{x_0} \otimes K(D)|_{x_0} \quad (4.6.6)$$

is surjective. By definition of the Hitchin map $h_r(\psi) \in H_{x_0}$ if and only if $\det(\varphi(x_0)) = 0$. On the other hand, $\varphi \in H^0(\mathrm{SPEnd}_0(E, E_\bullet) \otimes K(D - x_0))$ if and only if $\varphi(x_0) = 0$.

We will use the following algebra fact, $\psi(x_0) \in \mathrm{End}(E)|_{x_0} \otimes K(D)|_{x_0}$ is zero if and only if for every other matrix $M \in \mathrm{End}(E)|_{x_0} \otimes K(D)|_{x_0}$ such that $\det(M) = 0$, $\det(\psi(x_0) + M) = 0$. Finally, as the evaluation map (4.6.6) is surjective, the latter is equivalent to $\det(\psi(x_0) + \varphi(x_0)) = 0$ for every $\varphi \in h_r^{-1}(H_{x_0})$. \square

Lemma 4.6.15. *Suppose that $g \geq 4$. Let (E, E_\bullet) and (E', E'_\bullet) be generic parabolic vector bundles such that $\Phi(E, E_\bullet) = (E', E'_\bullet)$. Then Φ induces an isomorphism of vector bundles*

$$\Phi_{\text{SPEnd}_0} : \text{SPEnd}_0(E, E_\bullet) \cong \text{SPEnd}_0(E', E'_\bullet)$$

Proof. Let \mathcal{E} be the sub-bundle of the trivial vector bundle

$$H^0(\text{SPEnd}_0(E, E_\bullet) \otimes K(D)) \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow X$$

whose fiber over each $x \in X$ is $H^0(\text{SPEnd}_0(E, E_\bullet) \otimes K(D - x))$. From Lemma 4.1.7, the following sequence is exact

$$0 \rightarrow \mathcal{E} \rightarrow H^0(\text{SPEnd}_0(E, E_\bullet) \otimes K(D)) \otimes_{\mathbb{C}} \mathcal{O}_X \xrightarrow{\pi} \text{SPEnd}_0(E, E_\bullet) \otimes K(D) \rightarrow 0$$

where the last morphism is the evaluation map. Analogously, we define a vector bundle \mathcal{E}' such that

$$0 \rightarrow \mathcal{E}' \rightarrow H^0(\text{SPEnd}_0(E', E'_\bullet) \otimes K(D)) \otimes_{\mathbb{C}} \mathcal{O}_X \xrightarrow{\pi} \text{SPEnd}_0(E', E'_\bullet) \otimes K(D) \rightarrow 0$$

By Lemma 4.6.14, over $U = X \setminus D$, the image of $\mathcal{E}|_U$ under $d(\Phi^{-1}) \otimes \text{Id}_{\mathcal{O}_U}$ is $\mathcal{E}'|_U$. As \mathcal{E} and \mathcal{E}' are the saturations of $\mathcal{E}|_U$ and $\mathcal{E}'|_U$ in $H^0(\text{SPEnd}_0(E, E_\bullet) \otimes K(D)) \otimes_{\mathbb{C}} \mathcal{O}_U$ and $H^0(\text{SPEnd}_0(E', E'_\bullet) \otimes K(D)) \otimes_{\mathbb{C}} \mathcal{O}_U$ respectively, the image of \mathcal{E} under $d(\Phi^{-1}) \otimes \text{Id}_{\mathcal{O}_X}$ must be \mathcal{E}' . Therefore, passing to the quotient, there must exist an isomorphism of vector bundles

$$\Phi_{\text{SPEnd}_0} : \text{SPEnd}_0(E, E_\bullet) \cong \text{SPEnd}_0(E', E'_\bullet)$$

such that the following diagram commutes

$$\begin{array}{ccccc} 0 \rightarrow \mathcal{E} \rightarrow H^0(\text{SPEnd}_0(E, E_\bullet) \otimes K(D)) \otimes_{\mathbb{C}} \mathcal{O}_X & \xrightarrow{\pi} & \text{SPEnd}_0(E, E_\bullet) \otimes K(D) & \rightarrow & 0 \\ \downarrow & \downarrow d(\Phi^{-1}) \otimes \text{Id}_{\mathcal{O}_X} & \downarrow \Phi_{\text{SPEnd}_0} \otimes \text{Id}_{K(D)} & & \downarrow \\ 0 \rightarrow \mathcal{E}' \rightarrow H^0(\text{SPEnd}_0(E', E'_\bullet) \otimes K(D)) \otimes_{\mathbb{C}} \mathcal{O}_X & \xrightarrow{\pi} & \text{SPEnd}_0(E', E'_\bullet) \otimes K(D) & \rightarrow & 0 \end{array}$$

□

Lemma 4.6.16. *Suppose that $g \geq 6$. Let (E, E_\bullet) and (E', E'_\bullet) be generic parabolic vector bundles such that $\Phi(E, E_\bullet) = (E', E'_\bullet)$. Then Φ induces an isomorphism of vector bundles*

$$\Phi_{\text{PEnd}_0} : \text{PEnd}_0(E, E_\bullet) \cong \text{PEnd}_0(E', E'_\bullet)$$

such that the following diagram commutes

$$\begin{array}{ccc} \text{SPEnd}_0(E, E_\bullet) & \xrightarrow{\Phi_{\text{SPEnd}_0}} & \text{SPEnd}_0(E', E'_\bullet) \\ \downarrow & & \downarrow \\ \text{PEnd}_0(E, E_\bullet) & \xrightarrow{\Phi_{\text{PEnd}_0}} & \text{PEnd}_0(E', E'_\bullet) \end{array}$$

Proof. Given a parabolic point $x \in D$ and a parabolic vector bundle (E, E_\bullet) , let $\mathrm{SPEnd}_0^{(x)}(E, E_\bullet)$ be the subsheaf of $\mathrm{SPEnd}_0(E, E_\bullet)$ whose stalk over $y \in X \setminus \{x\}$ is $\mathrm{SPEnd}_0(E, E_\bullet)_y$ and whose stalk over x is $(\mathrm{PEnd}_0(E, E_\bullet)(-x))_x$. It fits into a short exact sequence

$$0 \longrightarrow \mathrm{SPEnd}_0^{(x)}(E, E_\bullet) \longrightarrow \mathrm{SPEnd}_0(E, E_\bullet) \longrightarrow \bigoplus_{k=1}^{r^2-r} k_x \longrightarrow 0$$

where the last morphism is the evaluation map at x of the elements of $\mathrm{SPEnd}_0(E, E_\bullet)$ out of the diagonal, once a basis compatible with the parabolic filtration is chosen. More explicitly, if $(E, \{E_{i,y}\})$ is the parabolic vector bundle obtained by restricting the parabolic filtration to $y \in D$, then we define

$$\mathrm{SPEnd}_0^{(x)}(E, E_\bullet) = \bigcap_{y \in D \setminus \{x\}} \mathrm{SPEnd}_0(E, \{E_{i,y}\}) \cap \mathrm{PEnd}_0(E, \{E_{i,x}\})$$

From the definition, it becomes clear that

$$\mathrm{SPEnd}_0(E, E_\bullet)(-x) \hookrightarrow \mathrm{SPEnd}_0^{(x)}(E, E_\bullet) \hookrightarrow \mathrm{SPEnd}_0(E, E_\bullet) \quad (4.6.7)$$

and these sheaves are related with $\mathrm{PEnd}_0(E, E_\bullet)$ by the following relation

$$\begin{aligned} \mathrm{SPEnd}_0(E, E_\bullet)(-D) \hookrightarrow \mathrm{PEnd}_0(E, E_\bullet)(-D) &= \\ &= \bigcap_{x \in D} \mathrm{SPEnd}_0^{(x)}(E, E_\bullet) \hookrightarrow \mathrm{SPEnd}_0(E, E_\bullet) \end{aligned} \quad (4.6.8)$$

We will prove that for every $x \in D$, Φ induces a morphism

$$\Phi_{\mathrm{SPEnd}_0}^{(x)} : \mathrm{SPEnd}_0^{(x)}(E, E_\bullet) \longrightarrow \mathrm{SPEnd}_0^{(x)}(E', E'_\bullet)$$

such that the following diagram commutes

$$\begin{array}{ccccc} \mathrm{SPEnd}_0(E, E_\bullet)(-x) & \hookrightarrow & \mathrm{SPEnd}_0^{(x)}(E, E_\bullet) & \hookrightarrow & \mathrm{SPEnd}_0(E, E_\bullet) \\ \downarrow \Phi_{\mathrm{SPEnd}_0} \otimes \mathrm{Id}_{\mathcal{O}_X(-x)} & & \downarrow \Phi_{\mathrm{SPEnd}_0}^{(x)} & & \downarrow \Phi_{\mathrm{SPEnd}_0} \\ \mathrm{SPEnd}_0(E', E'_\bullet)(-x) & \hookrightarrow & \mathrm{SPEnd}_0^{(x)}(E', E'_\bullet) & \hookrightarrow & \mathrm{SPEnd}_0(E', E'_\bullet) \end{array}$$

Then Φ_{SPEnd_0} preserves the subsheaf $\mathrm{SPEnd}_0^{(x)}(E, E_\bullet)$ and $\Phi_{\mathrm{SPEnd}_0}^{(x)}$ is simply the restriction of the morphism. Using the relation (4.6.8), we conclude that Φ_{SPEnd_0} preserves $\mathrm{PEnd}_0(E, E_\bullet)(-D)$, in the sense that it induces by restriction to the intersection a morphism

$$\overline{\Phi_{\mathrm{PEnd}_0}} : \mathrm{PEnd}_0(E, E_\bullet)(-D) \longrightarrow \mathrm{PEnd}_0(E', E'_\bullet)(-D)$$

such that the following diagram commutes

$$\begin{array}{ccccc} \mathrm{SPEnd}_0(E, E_\bullet)(-D) & \hookrightarrow & \mathrm{PEnd}_0(E, E_\bullet)(-D) & \hookrightarrow & \mathrm{SPEnd}_0(E, E_\bullet) \\ \downarrow \Phi_{\mathrm{SPEnd}_0} \otimes \mathrm{Id}_{\mathcal{O}_X(-D)} & & \downarrow \overline{\Phi_{\mathrm{PEnd}_0}} & & \downarrow \Phi_{\mathrm{SPEnd}_0} \\ \mathrm{SPEnd}_0(E', E'_\bullet)(-D) & \hookrightarrow & \mathrm{PEnd}_0(E', E'_\bullet)(-D) & \hookrightarrow & \mathrm{SPEnd}_0(E', E'_\bullet) \end{array}$$

finally, tensoring the previous diagram by $\mathcal{O}_X(D)$ and taking $\Phi_{\text{PEnd}_0} = \overline{\Phi_{\text{PEnd}_0}} \otimes \text{Id}_{\mathcal{O}_X(D)}$ yields the desired vector bundle isomorphism.

Now let us build the morphism $\Phi_{\text{SPEnd}_0}^{(x)}$. Let (E, E_\bullet) be a generic parabolic vector bundle. Let us define the following subsets of $H^0(\text{SPEnd}_0(E, E_\bullet) \otimes K(D)) = T_{(E, E_\bullet)}^* \mathcal{M}(r, \alpha, \xi)$ recursively.

$$\left\{ \begin{array}{ll} \forall k > 0 & F_{(E, E_\bullet)}^0 = H^0(\text{SPEnd}_0(E, E_\bullet) \otimes K(D)) \\ \forall k > 0 & G_{(E, E_\bullet)}^k = \{\psi \in F_{(E, E_\bullet)}^{k-1} : h_r(\psi) \in H^0(K^r D^{r-1}(-kx))\} \\ \forall k > 0, y \in X \setminus D & F_{(E, E_\bullet), y}^k = \{\psi \in G_{(E, E_\bullet)}^k : \forall \psi \in G_{(E, E_\bullet)}^k \varphi + \psi \in G_{(E, E_\bullet)}^k\} \\ \forall k > 0, y \in X \setminus D & G_{(E, E_\bullet), y}^k = \{\psi \in F_{(E, E_\bullet)}^k : h_r(\psi) \in H^0(K^r D^{r-1}(-kx - y))\} \\ \forall k > 0, y \in X \setminus D & F_{(E, E_\bullet), y}^k = \{\psi \in G_{(E, E_\bullet), y}^k : \forall \psi \in G_{(E, E_\bullet), y}^k \varphi + \psi \in G_{(E, E_\bullet), y}^k\} \end{array} \right. \quad (4.6.9)$$

By Lemma 4.6.13, f_r preserves $H^0(K^r D^{r-1}(-kx))$ for every k , so, by construction, for every $k > 0$

$$\begin{aligned} d(\Phi^{-1}) \left(F_{(E, E_\bullet)}^k \right) &= F_{(E', E'_\bullet)}^k \\ d(\Phi^{-1}) \left(G_{(E, E_\bullet)}^k \right) &= G_{(E', E'_\bullet)}^k \\ d(\Phi^{-1}) \left(F_{(E, E_\bullet), y}^k \right) &= F_{(E', E'_\bullet), y}^k \\ d(\Phi^{-1}) \left(G_{(E, E_\bullet), y}^k \right) &= G_{(E', E'_\bullet), y}^k \end{aligned}$$

We will prove the following equalities for $x \in D$ and $y \in X \setminus D$

$$\begin{aligned} F_{(E, E_\bullet)}^{r-1} &= H^0(\text{SPEnd}_0^{(x)}(E, E_\bullet) \otimes K(D)) \\ F_{(E, E_\bullet)}^r &= H^0(\text{SPEnd}_0(E, E_\bullet) \otimes K(D - x)) \\ F_{(E, E_\bullet), y}^{r-1} &= H^0(\text{SPEnd}_0^{(x)}(E, E_\bullet) \otimes K(D - y)) \\ F_{(E, E_\bullet), y}^r &= H^0(\text{SPEnd}_0(E, E_\bullet) \otimes K(D - x - y)) \end{aligned} \quad (4.6.10)$$

It is straightforward to test that

$$\left(\text{SPEnd}_0^{(x)}(E, E_\bullet) \otimes \mathcal{O}_X(D) \right)^\vee \cong \text{SPEnd}_0(E, \{E_{i,x}\})(x) \cap \text{PEnd}_0(E, E_\bullet)(x) \hookrightarrow \text{End}_0(E)(x)$$

As $g \geq 6$, Lemma 4.1.7 implies that for every $x \in D$ and every $y \in X$

$$H^1(\text{SPEnd}_0^{(x)}(E, E_\bullet) \otimes K(D - y)) = H^0 \left(\left(\text{SPEnd}_0^{(x)}(E, E_\bullet) \otimes \mathcal{O}_X(D) \right)^\vee \otimes \mathcal{O}_X(y) \right)^\vee = 0$$

$$H^1(\text{SPEnd}_0(E, E_\bullet) \otimes K(D - x - y)) = H^0(\text{PEnd}_0(E, E_\bullet)(x + y))^\vee = 0$$

Therefore, we have the following short exact sequences

$$\begin{array}{ccccccc} 0 \twoheadrightarrow H^0(\text{SPEnd}_0(E, E_\bullet) \otimes K(D - x - y)) & \twoheadrightarrow & H^0(\text{SPEnd}_0(E, E_\bullet) \otimes K(D - x)) & \twoheadrightarrow & \text{SPEnd}_0(E, E_\bullet) \otimes K(D - x)|_y & \twoheadrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \twoheadrightarrow H^0(\text{SPEnd}_0^{(x)}(E, E_\bullet) \otimes K(D - y)) & \longrightarrow & H^0(\text{SPEnd}_0^{(x)}(E, E_\bullet) \otimes K(D)) & \longrightarrow & \text{SPEnd}_0^{(x)}(E, E_\bullet) \otimes K(D)|_y & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \twoheadrightarrow H^0(\text{SPEnd}_0(E, E_\bullet) \otimes K(D - y)) & \longrightarrow & H^0(\text{SPEnd}_0(E, E_\bullet) \otimes K(D)) & \longrightarrow & \text{SPEnd}_0(E, E_\bullet) \otimes K(D)|_y & \longrightarrow & 0 \end{array}$$

which are reduced to the following diagram if $y \in X \setminus D$

$$\begin{array}{ccccccc}
0 \rightarrow & F_{(E, E_\bullet), y}^r & \rightarrow & F_{(E, E_\bullet)}^r & \rightarrow & \mathrm{SPEnd}_0(E, E_\bullet) \otimes K(D-x)|_y & \rightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \rightarrow & F_{(E, E_\bullet), y}^{r-1} & \rightarrow & F_{(E, E_\bullet)}^{r-1} & \rightarrow & \mathrm{SPEnd}_0^{(x)}(E, E_\bullet) \otimes K(D)|_y & \rightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \rightarrow & F_{(E, E_\bullet), y}^0 & \rightarrow & F_{(E, E_\bullet)}^0 & \rightarrow & \mathrm{SPEnd}_0(E, E_\bullet) \otimes K(D)|_y & \rightarrow 0
\end{array}$$

Let \mathcal{F} and \mathcal{G} be the sub-vector bundles

$$\mathcal{F} \hookrightarrow H^0(\mathrm{SPEnd}_0^{(x)}(E, E_\bullet) \otimes K(D-x)) \otimes \mathcal{O}_X$$

and

$$\mathcal{G} \hookrightarrow H^0(\mathrm{SPEnd}_0(E, E_\bullet) \otimes K(D-x)) \otimes \mathcal{O}_X$$

whose fiber over $y \in X$ is $H^0(\mathrm{SPEnd}_0^{(x)}(E, E_\bullet) \otimes K(D-y))$ and $H^0(\mathrm{SPEnd}_0(E, E_\bullet) \otimes K(D-x-y))$ respectively. We define the vector bundles \mathcal{F}' and \mathcal{G}' analogously for (E', E'_\bullet) . Then Lemma 4.1.7 implies that the rows of the following commutative diagram are exact

$$\begin{array}{ccccccc}
0 \rightarrow & \mathcal{G} & \rightarrow & F_{(E, E_\bullet)}^r \otimes \mathcal{O}_X & \rightarrow & \mathrm{SPEnd}_0(E, E_\bullet) \otimes K(D-x) & \rightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \rightarrow & \mathcal{F} & \rightarrow & F_{(E, E_\bullet)}^{r-1} \otimes \mathcal{O}_X & \rightarrow & \mathrm{SPEnd}_0^{(x)}(E, E_\bullet) \otimes K(D) & \rightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \rightarrow & \mathcal{E} & \rightarrow & F_{(E, E_\bullet)}^0 \otimes \mathcal{O}_X & \rightarrow & \mathrm{SPEnd}_0(E, E_\bullet) \otimes K(D) & \rightarrow 0
\end{array} \tag{4.6.11}$$

Over U , we have proven that

$$(d(\Phi^{-1}) \otimes \mathrm{Id}_{\mathcal{O}_U})(\mathcal{G}|_U) = \mathcal{G}'|_U$$

$$(d(\Phi^{-1}) \otimes \mathrm{Id}_{\mathcal{O}_U})(\mathcal{F}|_U) = \mathcal{F}'|_U$$

$$(d(\Phi^{-1}) \otimes \mathrm{Id}_{\mathcal{O}_U})(\mathcal{E}|_U) = \mathcal{E}'|_U$$

As before, \mathcal{G} , \mathcal{F} and \mathcal{E} are the saturations of $\mathcal{G}|_U$, $\mathcal{F}|_U$ and $\mathcal{E}|_U$ and the same holds for \mathcal{G}' , \mathcal{F}' and \mathcal{E}' , so

$$(d(\Phi^{-1}) \otimes \mathrm{Id}_{\mathcal{O}_X})(\mathcal{G}) = \mathcal{G}'$$

$$(d(\Phi^{-1}) \otimes \mathrm{Id}_{\mathcal{O}_X})(\mathcal{F}) = \mathcal{F}'$$

$$(d(\Phi^{-1}) \otimes \mathrm{Id}_{\mathcal{O}_X})(\mathcal{E}) = \mathcal{E}'$$

By commutativity of diagram (4.6.11), the morphisms between \mathcal{G} , \mathcal{F} and \mathcal{E} coincide with the restriction of the morphism $(d(\Phi^{-1}) \otimes \mathrm{Id}_{\mathcal{O}_X}) : F_{(E, E_\bullet)}^0 \otimes \mathcal{O}_X \rightarrow F_{(E, E_\bullet)}^0 \otimes \mathcal{O}_X$ to the corresponding subsheaves. Taking quotients, we obtain the following commutative diagram proving the desired result

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{G} & \longrightarrow & F''_{(E, E_\bullet)} \otimes \mathcal{O}_X & \longrightarrow & \mathrm{SPEnd}_0(E, E_\bullet) \otimes K(D-x) \longrightarrow 0 \\
& & \nearrow & & \nearrow d(\Phi^{-1}) \otimes \mathrm{Id}_{\mathcal{O}_X} & & \nearrow \Phi_{\mathrm{SPEnd}_0} \otimes \mathrm{Id}_{K(D-x)} \\
0 & \longrightarrow & \mathcal{G}' & \longrightarrow & F''_{(E', E'_\bullet)} \otimes \mathcal{O}_X & \longrightarrow & \mathrm{SPEnd}_0(E', E'_\bullet) \otimes K(D-x) \longrightarrow 0 \\
& & \nearrow & & \nearrow d(\Phi^{-1}) \otimes \mathrm{Id}_{\mathcal{O}_X} & & \nearrow \Phi_{\mathrm{SPEnd}_0}^{(x)} \otimes \mathrm{Id}_{K(D)} \\
0 & \longrightarrow & \mathcal{F} & \longrightarrow & F''_{(E, E_\bullet)} \otimes \mathcal{O}_X & \longrightarrow & \mathrm{SPEnd}_0^{(x)}(E, E_\bullet) \otimes K(D) \longrightarrow 0 \\
& & \nearrow & & \nearrow d(\Phi^{-1}) \otimes \mathrm{Id}_{\mathcal{O}_X} & & \nearrow \Phi_{\mathrm{SPEnd}_0}^{(x)} \otimes \mathrm{Id}_{K(D)} \\
0 & \longrightarrow & \mathcal{F}' & \longrightarrow & F''_{(E', E'_\bullet)} \otimes \mathcal{O}_X & \longrightarrow & \mathrm{SPEnd}_0^{(x)}(E', E'_\bullet) \otimes K(D) \longrightarrow 0 \\
& & \nearrow & & \nearrow d(\Phi^{-1}) \otimes \mathrm{Id}_{\mathcal{O}_X} & & \nearrow \Phi_{\mathrm{SPEnd}_0} \otimes \mathrm{Id}_{K(D)} \\
0 & \longrightarrow & \mathcal{E} & \longrightarrow & F^0_{(E, E_\bullet)} \otimes \mathcal{O}_X & \longrightarrow & \mathrm{SPEnd}_0(E, E_\bullet) \otimes K(D) \longrightarrow 0 \\
& & \nearrow & & \nearrow d(\Phi^{-1}) \otimes \mathrm{Id}_{\mathcal{O}_X} & & \nearrow \Phi_{\mathrm{SPEnd}_0} \otimes \mathrm{Id}_{K(D)} \\
0 & \longrightarrow & \mathcal{E}' & \longrightarrow & F^0_{(E, E_\bullet)} \otimes \mathcal{O}_X & \longrightarrow & \mathrm{SPEnd}_0(E', E'_\bullet) \otimes K(D) \longrightarrow 0
\end{array}$$

Finally, we have to prove the equalities in (4.6.10). Let us take the image of $F_{(E,E_\bullet)}^k$ and $G_{(E,E_\bullet)}^k$ by the evaluation map

$$\pi : H^0(\mathrm{SPEnd}_0(E, E_\bullet) \otimes K(D)) \twoheadrightarrow \mathrm{SPEnd}_0(E, E_\bullet) \otimes K(D)|_x$$

Let us identify the right hand side fiber with the vector space of traceless $n \times n$ complex matrices and let us define for $0 < k \leq r$

$$\begin{aligned} \overline{F_{(E,E_\bullet)}^k} &= \left\{ \psi = (\psi_{ij}) \in \mathrm{SPEnd}_0(E, E_\bullet) \otimes K(D)|_x : \begin{array}{l} \forall l \ 0 < l \leq k \ \forall (i, j) \in I^l \\ \psi_{ij} = 0 \end{array} \right\} \\ \overline{G_{(E,E_\bullet)}^k} &= \overline{F_{(E,E_\bullet)}^{k-1}} \cap \left\{ \psi = (\psi_{ij}) : \prod_{(i,j) \in I^k} \psi_{ij} = 0 \right\} \end{aligned}$$

where we take $\overline{F_{(E,E_\bullet)}^0} = \mathrm{SPEnd}_0(E, E_\bullet) \otimes K(D)|_x$ and

$$I^k = \{(i, j) \in [1, r]^2 : j - i \cong k \pmod{r}\}$$

By definition of $\mathrm{SPEnd}_0^{(x)}(E, E_\bullet)$, it is clear that the following identities hold

$$\begin{aligned} \pi^{-1} \left(\overline{F_{(E,E_\bullet)}^{r-1}} \right) &= H^0(\mathrm{SPEnd}_0^{(x)}(E, E_\bullet) \otimes K(D)) \\ \pi^{-1} \left(\overline{F_{(E,E_\bullet)}^r} \right) &= H^0(\mathrm{SPEnd}_0(E, E_\bullet) \otimes K(D - x)) \end{aligned}$$

Let us prove by induction that for every $0 < k \leq r$

$$\begin{aligned} \pi^{-1} \left(\overline{G_{(E,E_\bullet)}^k} \right) &= G_{(E,E_\bullet)}^k \\ \pi^{-1} \left(\overline{F_{(E,E_\bullet)}^k} \right) &= F_{(E,E_\bullet)}^k \end{aligned}$$

For $k = 0$ the statement is trivial by construction. Suppose that

$$\pi^{-1} \left(\overline{F_{(E,E_\bullet)}^{k-1}} \right) = F_{(E,E_\bullet)}^{k-1}$$

Let $s \in H^0(\mathrm{SPEnd}_0(E, E_\bullet) \otimes K(D))$ be a section in $F_{(E, E_\bullet)}^{k-1}$. Let s_x be its germ at x . Looking at the image of the germ in $(\mathrm{End}_0(E) \otimes K(D))_x$, it can be identified with a matrix $S = (S_{ij}) \in \mathrm{Mat}_{n \times n}(\mathcal{O}_{X,x})$. As $\mathcal{O}_{X,x}$ is a local principal ideal domain, there exists an element $z \in \mathcal{O}_{X,x}$ such that $(x) \subset \mathcal{O}_{X,x}$ is the maximal ideal. For $-r < k < r$, let us denote by

$$D^k = \{(i, j) \in [1, r]^2 : j - i = k\}$$

the set of indexes corresponding to the k -th secondary diagonal of an $n \times n$ matrix. Note that for $0 < k < r$

$$I^k = D^k \cup D^{k-r}$$

and for $k = r$, $I^r = D^0$. By induction hypothesis, as $s \in F_{(E, E_\bullet)}^{k-1}$, then $z|S_{ij}$ for each $(i, j) \in D^l$ for $0 \leq l < k$ and, moreover, $z^2|S_{ij}$ for each $(i, j) \in D^{l-r}$ for $0 < l < k$. We have that $h_r(s) \in H^0(K^r D^{r-1}(-kx))$ if and only if $z^{k+1}|\det(S)$. Developing the determinant

$$\det(S) = \sum_{\sigma \in \Sigma_r} (-1)^{|\sigma|} \prod_{i=1}^r S_{i\sigma(i)}$$

the only summand with less than $k+1$ factors z is the product of the elements in I^k . To check this, observe that the only factors not already divisible by z are those with $j \geq i+k$. Moreover, note that for $i > r-k$, all the elements S_{ij} with $j < i+k-r$ are divisible by z^2 . Therefore, a permutation $\sigma : [1, r] \rightarrow [1, r]$ for which $\prod_{i=1}^r S_{i\sigma(i)}$ is not already divisible by z^{k+1} must have

1. $\sigma(i) \geq i+k$ for every $i \leq r-k$
2. $\sigma(i) \geq i+k-r$ for every $i > r-k$

Now the result follows from Lemma 4.6.17. Therefore, $z^{k+1}|\det(S)$ if and only if

$$z^{k+1} | \prod_{(i,j) \in I^k} S_{ij}$$

As the k elements below the diagonal are already multiple of z , the product is a multiple of z^{k+1} if and only if there is at least an extra z factor in some of the S_{ij} , i.e., if and only if s_{ij} annihilates for some $(i, j) \in I^k$. Therefore, taking into account that π is surjective, we obtain that $s \in G_{(E, E_\bullet)}^k$ if and only if $\pi(x) \in \overline{G_{(E, E_\bullet)}^k}$.

Now, let us prove that

$$\overline{F_{(E, E_\bullet)}^k} = \left\{ \psi \in \overline{G_{(E, E_\bullet)}^k} : \forall \varphi \in \overline{G_{(E, E_\bullet)}^k} \psi + \varphi \in \overline{G_{(E, E_\bullet)}^k} \right\}$$

Suppose that an element $\psi \in \overline{G_{(E, E_\bullet)}^k}$ has some $(i, j) \in I^k$ with $\psi_{ij} \neq 0$. Let $\emptyset \neq \mathcal{I} \subsetneq I^k$ be the subset of indexes in I^k such that $\psi_{ij} \neq 0$. Then, let us define $\varphi \in \overline{G_{(E, E_\bullet)}^k}$ in the following way

$$\varphi_{ij} = \begin{cases} 0 & (i, j) \in \mathcal{I} \\ 1 & (i, j) \in I^k \setminus \mathcal{I} \\ \psi_{ij} & (i, j) \in [1, n]^2 \setminus I^k \end{cases}$$

We can test that as $\psi \in \overline{F_{(E, E_\bullet)}^{k-1}}$, $\varphi \in \overline{F_{(E, E_\bullet)}^{k-1}}$ and as $\mathcal{I} \neq \emptyset$, then $\prod_{(i,j) \in I^k} \varphi_{ij} = 0$. On the other hand, for every $(i, j) \in I^k$

$$(\psi + \varphi)_{ij} \neq 0$$

so $\varphi + \psi \notin \overline{G_{(E, E_\bullet)}^k}$. Now the equality

$$\begin{aligned} \pi \left(\overline{F_{(E, E_\bullet)}^k} \right) &= \pi \left(\left\{ \psi \in G_{(E, E_\bullet)}^k : \forall \varphi \in G_{(E, E_\bullet)}^k \psi + \varphi \in G_{(E, E_\bullet)}^k \right\} \right) \\ &= \left\{ \psi \in \overline{G_{(E, E_\bullet)}^k} : \forall \varphi \in \overline{G_{(E, E_\bullet)}^k} \psi + \varphi \in \overline{G_{(E, E_\bullet)}^k} \right\} = \overline{F_{(E, E_\bullet)}^k} \end{aligned}$$

follows from surjectivity of $\pi : G_{(E, E_\bullet)}^k \rightarrow \overline{G_{(E, E_\bullet)}^k}$. The remaining equalities of (4.6.10)

$$\begin{aligned} F_{(E, E_\bullet), y}^{r-1} &= H^0(\mathrm{SPEnd}_0^{(x)}(E, E_\bullet) \otimes K(D - y)) \\ F_{(E, E_\bullet), y}^r &= H^0(\mathrm{SPEnd}_0(E, E_\bullet) \otimes K(D - x - y)) \end{aligned}$$

follow from the given ones using the same argument as the one used for Lemma 4.6.15, taking into account that, as we have already proven, the assumption $g \geq 6$ implies that the following morphisms are surjective for every $y \in X \setminus D$

$$\begin{aligned} H^0(\mathrm{SPEnd}_0(E, E_\bullet) \otimes K(D - x)) &\twoheadrightarrow \mathrm{SPEnd}_0(E, E_\bullet) \otimes K(D - x)|_y \\ H^0(\mathrm{SPEnd}_0^{(x)}(E, E_\bullet) \otimes K(D)) &\twoheadrightarrow \mathrm{SPEnd}_0^{(x)}(E, E_\bullet) \otimes K(D)|_y \end{aligned}$$

□

Lemma 4.6.17. *Let $\sigma : [1, r] \rightarrow [1, r]$ be a permutation such that*

1. $\sigma(i) \geq i + k$ for every $i \leq r - k$
2. $\sigma(i) \geq i + k - r$ for every $i > r - k$

Then

$$\sigma(i) = \begin{cases} i + k & i \leq r - k \\ i + k - r & i > r - k \end{cases}$$

Proof. Let us prove by induction that $\sigma(i) \leq i + k$ for $i \leq r - k$. For $i = r - k$, we have that

$$\sigma(r - k) \leq r = r - k + k$$

Let us assume that it is true for all $r - k \geq i > j$. Then $\sigma(i) = i + k$ for $r \geq i > j$. Therefore, the elements $[j + k + 1, r]$ have been selected by the permutation, so $\sigma(j) \notin [j + k + 1, r]$, so $\sigma(j) \leq j + k$. Once we know that $\sigma(i) = i + k$ for $i \leq r - k$, let us prove by induction that $\sigma(i) \leq i + k - r$ for every $i > r - k$. As the elements $[k + 1, r]$ have already been selected by the permutation, we know that $\sigma(i) \in [i + k - r, k]$ for every $i > r - k$. For $i = r$, we have $\sigma(r) \leq k = r + k - r$. Let $j > r - k$ and suppose that it is true for every $r \geq i > j$. Then $\sigma(i) = i + k - r$ for every $i > j$. Therefore, the elements $[j + k - r + 1, k]$ have already been selected by the permutation and we get $\sigma(j) \leq j + k - r$. □

Lemma 4.6.18. *Suppose that $g \geq 4$. For every $x \in X$, and every $k > 1$, the linear subspace*

$$H^0(K^k D^{k-1}(-kx)) \subseteq W_k$$

is preserved by the linear map $f_k : W_k \rightarrow W_k$.

Proof. Let $\mathcal{U} \subset \mathcal{M}(r, \alpha, \xi)$ and $\mathcal{U}' \subset \mathcal{M}(r, \alpha', \xi')$ be the open nonempty subsets of generic parabolic vector bundles in the sense of Lemma 4.1.7. Let $\mathcal{V} = \Phi^{-1}(\mathcal{U}) \cap \mathcal{U}'$ and let $\mathcal{V}' = \Phi(\mathcal{V}) \subseteq \mathcal{U}'$. They are also nonempty open subsets of $\mathcal{M}(r, \alpha, \xi)$ and $\mathcal{M}(r, \alpha', \xi')$ respectively. As we assumed $g \geq 3$, we have

$$r \deg(K(D - x)) = r(2g - 3 + n) \geq r(2g - 2) \geq 2(2g - 2) > 2g$$

Therefore, we can apply Corollary 4.2.3 to $L = K(D - x)$ and the open subsets U and U'' . Then we obtain that the linear subspace

$$\bigoplus_{k=2}^r H^0(K^k D^{k-1}(-kx)) \subseteq W$$

is the space generated by the images $h(H^0(\mathrm{SPEnd}_0(E, E_\bullet) \otimes K(D - x)))$ both when (E, E_\bullet) runs over \mathcal{V} and when (E, E_\bullet) runs over \mathcal{V}' .

By Lemma 4.6.15, for every $(E, E_\bullet) \in \mathcal{V}$, if $(E', E'_\bullet) = \Phi(E, E_\bullet) \in \mathcal{V}'$, then the image of $H^0(\mathrm{SPEnd}_0(E, E_\bullet) \otimes K(D - x))$ by $d(\Phi^{-1})$ is $H^0(\mathrm{SPEnd}_0(E', E'_\bullet) \otimes K(D - x))$. As $\Phi(\mathcal{V}) = \mathcal{V}'$, then union of the images $h(H^0(\mathrm{SPEnd}_0(E, E_\bullet) \otimes K(D - x)))$ for $(E, E_\bullet) \in \mathcal{V}$ is the same as the union of the images $h(H^0(\mathrm{SPEnd}_0(E', E'_\bullet) \otimes K(D - x)))$ for $(E', E'_\bullet) \in \mathcal{V}'$, so $f : W \rightarrow W$ preserves the subspace $\bigoplus_{k=2}^r H^0(X, K^k D^{k-1}(-kx)) \subseteq W$.

Finally, the result follows as a consequence of Lemma 4.6.11, as the map $f : W \rightarrow W$ is diagonal with respect to the decomposition $W = \bigoplus_{k=2}^r W_k$. \square

For $k > 1$, the curve X is embedded in $\mathbb{P}(W_k)$ via the linear system $|K^k D^{k-1}|$ and the osculating k -space at each point $x \in X$ is given by

$$\mathrm{Osc}_k(x) = \mathbb{P} \left(\ker \left(H^0(K^k D^{k-1})^\vee \rightarrow H^0(K^k D^{k-1}(-kx))^\vee \right) \setminus \{0\} \right)$$

The previous corollary, together with Lemma 4.6.15 proves that the morphism

$$\mathbb{P}(f_k) : \mathbb{P} \left(H^0(K^k D^{k-1})^\vee \setminus \{0\} \right) \longrightarrow \mathbb{P} \left(H^0(K^k D^{k-1})^\vee \setminus \{0\} \right)$$

preserves $\mathrm{Osc}_k(x)$ for all $x \in X$. Now, we use the following Lemma

Lemma 4.6.19. *Let $X \hookrightarrow \mathbb{P}^N$ be an irreducible smooth complex projective curve embedded in the projective space. If $\varphi \in \mathrm{PGL}(N + 1)$ is an isomorphism preserving $\mathrm{Osc}_k : X \rightarrow \mathrm{Gr}(k + 1, N + 1)$ for some k , then it preserves $\mathrm{Osc}_k : X \rightarrow \mathrm{Gr}(k + 1, N + 1)$ for every k .*

Proof. This is a direct consequence of the following fact proved in [BGM12, p. 1250052-23]. Let $X \hookrightarrow \mathbb{P}^N$ be an embedding of an irreducible smooth complex projective curve X in a projective space and let $\mathrm{Osc}_k : X \rightarrow \mathrm{Gr}(k + 1, N + 1)$ be the map sending each $x \in X$ to the osculating k -space of X in \mathbb{P}^N . Then Osc_k uniquely determines the embedding $X \hookrightarrow \mathbb{P}^N$. \square

As $\mathbb{P}(f_k)$ preserves Osc_k , it preserves Osc_1 , so f_k must preserve the hyperplanes

$$H^0(K^k D^{k-1}(-x)) \subset H^0(K^k D^{k-1})$$

for every $x \in X$.

In particular, this implies that for every $x \in X$ and generic (E, E_\bullet) the image of the set

$$N_{E,x} = \{\psi \in H^0(\text{SPEnd}_0(E, E_\bullet) \otimes K(D)) : \forall k > 1 \ h_k(\psi) \in H^0(K^k D^{k-1}(-x))\}$$

by $d(\Phi^{-1}) = H^0(\Phi_{\text{SPEnd}_0} \otimes \text{Id})$ is

$$N_{E',x} = \{\psi \in H^0(\text{SPEnd}_0(E', E'_\bullet) \otimes K(D)) : \forall k > 1 \ h_k(\psi) \in H^0(K^k D^{k-1}(-x))\}$$

For $x \in U$, the set $N_{E,x}$ coincides with the preimage of the nilpotent cone under the surjective map

$$H^0(\text{SPEnd}_0(E, E_\bullet) \otimes K(D)) \twoheadrightarrow \text{SPEnd}_0(E, E_\bullet) \otimes K(D)|_x$$

Taking the image of $N_{E,x}$ under the evaluation map we get a subset $\mathcal{N}_{E,x} \subset \text{SPEnd}_0(E, E_\bullet) \otimes K(D)|_x$. Varying x over U , we get a subscheme

$$\mathcal{N}_E|_U \hookrightarrow \text{SPEnd}_0(E, E_\bullet)|_U$$

such that $\Phi_{\text{SPEnd}_0}|_U(\mathcal{N}_E|_U) = \mathcal{N}_{E'}|_U$.

Therefore, if $g \geq 6$, $\Phi_{\text{PEnd}_0}|_U : \text{PEnd}_0(E, E_\bullet)|_U \rightarrow \text{PEnd}_0(E', E'_\bullet)|_U$ is an isomorphism of vector bundles that preserves the nilpotent cone. Therefore, it is an isomorphism of $\text{GL}(\text{par}\mathfrak{sl})|_U \cong \text{GL}(\mathfrak{sl}) \times U$ torsors that preserves the nilpotent cone. Let $N < \mathfrak{sl}$ denote the subalgebra of nilpotent matrices. Then, let us denote by

$$G_N = \{g \in \text{GL}(\mathfrak{sl}) : g(N) = N\} < \text{GL}(\mathfrak{sl})$$

the subgroup of invertible linear transformations of \mathfrak{sl} which preserve the nilpotent matrices. As $\Phi_{\text{PEnd}_0}|_U$ preserves the nilpotent cone, it is an isomorphism of G_N -torsors. Now, we can use the following theorem from Botta, Pierce and Watkins [BPW83],

Theorem 4.6.20. *The group G_N is generated by*

1. *Inner automorphisms $X \mapsto S^{-1}XS$*
2. *The maps $X \mapsto aX$ for some $a \neq 0$*
3. *The map $X \mapsto X^t$ that sends a matrix X to its transpose*

Using the computation in [BGM13, Lemma 5.4], we know that $\text{Aut}(\mathfrak{sl})$ is generated by inner automorphisms and the map $X \mapsto -X^t$. Therefore, we conclude that $G_N \cong \text{Aut}(\mathfrak{sl}) \times \mathbb{C}^*$. Thus, up to product by a morphism $U \rightarrow \mathbb{C}^*$, $\Phi_{\text{PEnd}_0}|_U$ is an isomorphism of $\text{Aut}(\mathfrak{sl})$ -torsors, i.e., it is an automorphism of Lie algebra bundles.

Lemma 4.6.21. *Suppose that $g \geq 6$. Let (E, E_\bullet) and (E', E'_\bullet) be generic parabolic vector bundles such that $\Phi(E, E_\bullet) = (E', E'_\bullet)$. Then there exists a constant $\lambda \in \mathbb{C}^*$ such that the vector bundle isomorphism $\lambda \cdot \Phi_{\text{PEnd}_0}$ defined in Lemma 4.6.16 is an isomorphism of Lie algebras bundles.*

Proof. As $\mathrm{PEnd}_0(E, E_\bullet)$ and $\mathrm{PEnd}_0(E', E'_\bullet)$ have the same degree, $\det(\Phi_{\mathrm{PEnd}_0}) \in H^0(X, \mathcal{O}_X)$. X is projective and connected, so $\det(\Phi_{\mathrm{PEnd}_0})$ is constant. The previous discussion shows that $\Phi_{\mathrm{PEnd}_0}|_U$ is an isomorphism of $(\mathrm{Aut}(\mathfrak{sl}) \times \mathbb{C})$ -torsors. As its determinant is constant, there exists a nonzero $\lambda \in \mathbb{C}^*$ such that $\lambda \cdot \Phi_{\mathrm{PEnd}_0}|_U$ is an isomorphism of $\mathrm{Aut}(\mathfrak{sl})$ -torsors, i.e., it is an isomorphism of Lie algebra bundles.

A Lie algebra structure on $\mathrm{PEnd}_0(E, E_\bullet)$ is in particular a bilinear morphism

$$[\cdot, \cdot] : \mathrm{PEnd}_0(E, E_\bullet) \otimes \mathrm{PEnd}_0(E, E_\bullet) \longrightarrow \mathrm{PEnd}_0(E, E_\bullet)$$

Therefore, the Lie algebra structure induced by endomorphism composition on (E, E_\bullet) is represented by a section

$$p_{(E, E_\bullet)} \in H^0(\mathrm{PEnd}_0(E, E_\bullet)^\vee \otimes \mathrm{PEnd}_0(E, E_\bullet)^\vee \otimes \mathrm{PEnd}_0(E, E_\bullet))$$

Similarly, the Lie algebra structure on (E', E'_\bullet) is represented by a section

$$p_{(E', E'_\bullet)} \in H^0(\mathrm{PEnd}_0(E', E'_\bullet)^\vee \otimes \mathrm{PEnd}_0(E', E'_\bullet)^\vee \otimes \mathrm{PEnd}_0(E', E'_\bullet))$$

Through the isomorphism $\lambda \cdot \Phi_{\mathrm{PEnd}_0}$, the section $p_{(E', E'_\bullet)}$ induces another section

$$(\lambda \cdot \Phi_{\mathrm{PEnd}_0})^* p_{(E', E'_\bullet)} \in H^0(\mathrm{PEnd}_0(E, E_\bullet)^\vee \otimes \mathrm{PEnd}_0(E, E_\bullet)^\vee \otimes \mathrm{PEnd}_0(E, E_\bullet))$$

Therefore, we obtain a section $p_{(E, E_\bullet)} - (\lambda \cdot \Phi_{\mathrm{PEnd}_0})^* p_{(E', E'_\bullet)}$. As $\lambda \cdot \Phi_{\mathrm{PEnd}_0}|_U$ is an isomorphism of Lie algebra sheaves, we obtain that $(p_{(E, E_\bullet)} - (\lambda \cdot \Phi_{\mathrm{PEnd}_0})^* p_{(E', E'_\bullet)})|_U = 0$, so $p_{(E, E_\bullet)} - (\lambda \cdot \Phi_{\mathrm{PEnd}_0})^* p_{(E', E'_\bullet)} = 0$ and $\lambda \cdot \Phi_{\mathrm{PEnd}_0}$ must be an isomorphism of Lie algebras. \square

Theorem 4.6.22. *Let (X, D) and (X', D') be two smooth projective curves of genus $g \geq 6$ and $g' \geq 6$ respectively with set of marked points $D \subset X$ and $D' \subset X'$. Let ξ and ξ' be line bundles over X and X' respectively, and let α and α' be full flag generic systems of weights over (X, D) and (X', D') respectively. Let*

$$\Phi : \mathcal{M}(X, r, \alpha, \xi) \xrightarrow{\sim} \mathcal{M}(X', r', \alpha', \xi')$$

be an isomorphism. Then

1. $r = r'$
2. (X, D) is isomorphic to (X', D') , i.e., there exists an isomorphism $\sigma : X \xrightarrow{\sim} X'$ sending D to D' .
3. There exists a basic transformation T such that
 - $\sigma^* \xi' \cong T(\xi)$
 - $\sigma^* \alpha'$ is in the same stability chamber as $T(\alpha)$.
 - For every $(E, E_\bullet) \in \mathcal{M}(r, \alpha, \xi)$, $\sigma^* \Phi(E, E_\bullet) \cong T(E, E_\bullet)$

Proof. Let $\Phi : \mathcal{M}(X, r, \alpha, \xi) \longrightarrow \mathcal{M}(X', r', \alpha', \xi')$ be an isomorphism. By Torelli Theorem 4.3.6, we obtain that $r = r'$ and there exists an isomorphism $\sigma : (X, D) \xrightarrow{\sim} (X', D')$. Pulling back by that isomorphism, we obtain an isomorphism

$$\Phi' : \mathcal{M}(X, r, \alpha, \xi) \longrightarrow \mathcal{M}(X, r, \sigma^* \alpha', \sigma^* \xi)$$

From this point, all the moduli spaces will be constructed over the same curve (X, D) , so, in order to simplify the notation, from now on, we will denote $\mathcal{M}(r, \alpha, \xi) = \mathcal{M}(X, r, \alpha, \xi)$. Let $\xi'' = \sigma^* \xi'$ and $\alpha'' = \sigma^* \alpha'$. The differential of Φ' induces an isomorphism of the cotangent bundles $d(\Phi')^{-1} : T^* \mathcal{M}(r, \alpha, \xi) \rightarrow T^* \mathcal{M}(r, \alpha'', \xi'')$. Let $h : T^* \mathcal{M}(r, \alpha, \xi) \rightarrow W$ and $h'' : T^* \mathcal{M}(r, \alpha'', \xi'') \rightarrow W$ denote the Hitchin morphisms corresponding to each choice of the system of weights and determinant. Since both moduli spaces are built over the same marked curve (X, D) for the same rank, the Hitchin space is the same for both moduli spaces. By Proposition 4.3.4, there exists a \mathbb{C}^* -equivariant automorphism $f : W \rightarrow W$ such that the following diagram commutes

$$\begin{array}{ccc} T^* \mathcal{M}(r, \alpha, \xi) & \xrightarrow{d(\Phi')^{-1}} & T^* \mathcal{M}(r, \alpha'', \xi'') \\ h \downarrow & & \downarrow h'' \\ W & \xrightarrow{f} & W \end{array}$$

As f is \mathbb{C}^* -equivariant, it preserves the subspace of maximum decay $W_r \subset W$. Let $h_r : T^* \mathcal{M}(r, \alpha, \xi) \rightarrow W_r$ (respectively $h''_r : T^* \mathcal{M}(r, \alpha'', \xi'') \rightarrow W_r$) be the composition of h with the projection to W_r . Let $f_r : W_r \rightarrow W_r$ be the restriction of f to W_r . Then f_r is linear and, by Corollary 4.6.12 we have a diagram

$$\begin{array}{ccc} T^* \mathcal{M}(r, \alpha, \xi) & \xrightarrow{d(\Phi')^{-1}} & T^* \mathcal{M}(r, \alpha'', \xi'') \\ h_r \downarrow & & \downarrow h''_r \\ W_r & \xrightarrow{f_r} & W_r \end{array}$$

By Lemma 4.6.13 for every $k > 0$ and every $x_0 \in X$

$$f_r(H^0(K^r D^{r-1}(-kx_0))) = H^0(K^r D^{r-1}(-kx_0))$$

By Corollary 4.1.4 and Lemma 4.1.7, there exists an open nonempty subset $\mathcal{U} \subseteq \mathcal{M}(r, \alpha, \xi)$ (respectively $\mathcal{U}'' \subseteq \mathcal{M}(r, \alpha'', \xi'')$) parameterizing α'' -stable (respectively α -stable) parabolic vector bundles (E, E_\bullet) such that

$$H^1(\mathrm{SPend}_0(E, E_\bullet) \otimes K(D - x - y)) = 0$$

for every $x, y \in X$. Let $\mathcal{V} = \mathcal{U} \cap (\Phi')^{-1}(\mathcal{U}'')$ and $\mathcal{V}'' = \Phi'(\mathcal{V})$. By definition of \mathcal{V}'' , there is a natural identification between \mathcal{V}'' and an open nonempty subset in $\mathcal{M}(r, \alpha, \xi'')$. Let $(E, E_\bullet) \in \mathcal{M}(r, \alpha, \xi)$ and let $\Phi'(E, E_\bullet) = (E'', E''_\bullet) \in \mathcal{M}(r, \alpha, \xi'')$ be its image. Therefore, we can apply Lemma 4.6.21 and we obtain that $\mathrm{PEnd}_0(E, E_\bullet)$ is isomorphic to $\mathrm{PEnd}_0(E'', E''_\bullet)$ as Lie algebra bundles. Then Lemma 4.5.14 proves that (E', E'_\bullet) can be obtained from (E, E_\bullet) as a combination of the following transformations

1. Tensorization with a line bundle over X , $(E, E_\bullet) \mapsto (E \otimes L, E_\bullet \otimes L)$
2. Dualization $(E, E_\bullet) \mapsto (E, E_\bullet)^\vee$
3. Hecke transformation at a parabolic point $x \in D$, $(E, E_\bullet) \mapsto \mathcal{H}_x(E, E_\bullet)$.

This means that $(E'', E_\bullet'') = T(E, E_\bullet)$ for some basic transformation $T = (\text{Id}, s, L, H)$. In particular, we obtain that $\xi'' = T(\xi)$. As the set of possible values for H in the choice of T is finite and the r -torsion of the Jacobian $J(X)$ is finite, the space of basic transformations

$$\mathcal{T}_{\xi, \xi''} = \{T = (\text{Id}, s, L, H) \in \mathcal{T} \mid T(\xi) \cong \xi''\}$$

is finite. For every $T \in \mathcal{T}_{\xi, \xi''}$, let us consider the composition of isomorphisms $T \circ (\Phi')^{-1} : \mathcal{M}(r, \alpha'', \xi'') \rightarrow \mathcal{M}(r, T(\alpha), \xi'')$. By construction of \mathcal{V} and \mathcal{V}' , it sends \mathcal{V}'' to $T(\mathcal{V})$

$$\begin{array}{ccccc} \mathcal{M}(r, \alpha'', \xi'') & \xrightarrow{(\Phi')^{-1}} & \mathcal{M}(r, \alpha, \xi) & \xrightarrow{T} & \mathcal{M}(r, T(\alpha), \xi'') \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{V}'' & \longrightarrow & \mathcal{V} & \longrightarrow & T(\mathcal{V}) \end{array}$$

Both \mathcal{V} and $T(\mathcal{V})$ parameterize parabolic vector bundles of rank r and determinant ξ which are both α -semistable and α'' -semistable and are generic in the sense of Lemma 4.1.7, so they can be canonically identified. Choose once and for all an identification $\mathcal{V}'' \cong T(\mathcal{V})$. Let $\Psi_T : \mathcal{V}'' \rightarrow \mathcal{V}''$ be the automorphism of \mathcal{V}'' induced composing $T \circ (\Phi')^{-1}$ with the identification $\mathcal{V}'' \cong T(\mathcal{V})$.

For every $(E, E_\bullet) \in \mathcal{V}$ there exists some $T \in \mathcal{T}_{\xi, \xi''}$ such that $\Phi'(E, E_\bullet) = T(E, E_\bullet)$. Therefore, for every $(E'', E_\bullet'') \in \mathcal{V}''$ there exists some $T \in \mathcal{T}_{\xi, \xi''}$ such that $\Psi_T(E'', E_\bullet'') = (E'', E_\bullet'')$ and we obtain that

$$\mathcal{V}'' = \bigcup_{T \in \mathcal{T}_{\xi, \xi''}} \text{Fix}(\Psi_T)$$

As the set of fixed points of an automorphism is closed and $\mathcal{T}_{\xi, \xi''}$ is finite, \mathcal{V}'' is a finite union of closed subsets. $\mathcal{M}(r, \alpha'', \xi'')$ is irreducible and \mathcal{V}'' is open, so \mathcal{V}'' is irreducible. Then there exists some $T \in \mathcal{T}_{\xi, \xi''}$ such that $\mathcal{V}'' = \text{Fix}(\Psi_T)$. Therefore, we conclude that there exist $T \in \mathcal{T}_{\xi, \xi''}$ and an open subset $\mathcal{V} \subseteq \mathcal{M}(r, \alpha, \xi)$ such that $\Phi'|_{\mathcal{V}} = T|_{\mathcal{V}}$.

Let us prove that, in fact, we can find an open subset $\mathcal{W} \subseteq \mathcal{M}(r, \alpha, \xi)$ whose complement has codimension at least 2 and such that $\Phi'|_{\mathcal{V}} = T|_{\mathcal{V}}$. Let $\mathcal{W} \subset \mathcal{M}(r, \alpha, \xi)$ be the space of parabolic vector bundles which are both α -stable and $T^{-1}(\alpha'')$ -stable. By Corollary 4.1.4, the complement of \mathcal{W} has codimension at least 2. Clearly, T is well defined over \mathcal{W} and it gives us a map $T : \mathcal{W} \rightarrow \mathcal{M}(r, \alpha'', \xi'')$. Moreover, as $\mathcal{M}(r, \alpha, \xi)$ is irreducible, $\mathcal{W} \cap \mathcal{V}$ is dense in \mathcal{W} , so every map $\psi : \mathcal{W} \cap \mathcal{V} \rightarrow \mathcal{M}(r, \alpha'', \xi'')$ admits a unique extension to \mathcal{W} by continuity. We know that $\Phi'|_{\mathcal{V} \cap \mathcal{W}} = T|_{\mathcal{V} \cap \mathcal{W}}$, and $\Phi'|_{\mathcal{W}}$ and $T|_{\mathcal{W}}$ are two possible extensions, so they must coincide.

As α'' is a full flag system of weights, $\mathcal{M}(r, \alpha'', \xi'')$ is a fine moduli space for every ξ'' . Therefore, Φ' is represented by a parabolic vector bundle $(\mathcal{E}'', \mathcal{E}_\bullet'')$ over $\mathcal{M}(r, \alpha, \xi) \times X$ whose fibers are α'' -stable as parabolic vector bundles over X . We

have the following commutative diagram

$$\begin{array}{ccc}
 \Phi' \vdash & \xrightarrow{\quad} & (\mathcal{E}'', \mathcal{E}_\bullet'') \\
 \downarrow & & \downarrow \\
 \text{Hom}(\mathcal{M}(r, \alpha, \xi), \mathcal{M}(r, \alpha'', \xi'')) & \xrightarrow{\sim} & \mathcal{M}(r, \alpha'', \xi'')(\mathcal{M}(r, \alpha, \xi)) \\
 \downarrow i^\# & & \downarrow i^* \\
 \text{Hom}(\mathcal{W}, \mathcal{M}(r, \alpha'', \xi'')) & \xrightarrow{\sim} & \mathcal{M}(r, \alpha'', \xi'')(\mathcal{W}) \\
 \downarrow & & \downarrow \\
 T \vdash & \xrightarrow{\quad} & T(\mathcal{E}, \mathcal{E}_\bullet)|_{\mathcal{W}}
 \end{array}$$

where $(\mathcal{E}, \mathcal{E}_\bullet)$ is the universal family of the moduli space $\mathcal{M}(r, \alpha, \xi)$. Therefore, $(\mathcal{E}'', \mathcal{E}_\bullet'')$ is the extension of the basic transformation $T(\mathcal{E}, \mathcal{E}_\bullet)|_{\mathcal{W}}$ from \mathcal{W} to all the moduli space. Note that, $T(\mathcal{E}, \mathcal{E}_\bullet)$ is a possible extension as a family of parabolic vector bundles over $\mathcal{M}(r, \alpha, \xi)$. By construction, we know that the codimension of the complement of \mathcal{W} in $\mathcal{M}(r, \alpha, \xi)$ is at least 2 and $\mathcal{M}(r, \alpha, \xi)$ is a smooth complex projective scheme, so by Lemma 4.1.8

$$(\mathcal{E}'', \mathcal{E}_\bullet'') \cong T(\mathcal{E}, \mathcal{E}_\bullet)$$

As $(\mathcal{E}'', \mathcal{E}_\bullet'')$ is a family of α'' -stable vector bundles, we conclude that $T(\mathcal{E}, \mathcal{E}_\bullet)$ is a family of α'' -stable vector bundles. Nevertheless, it is also a universal family of $T(\alpha)$ -stable vector bundles. We know that $(\mathcal{E}'', \mathcal{E}_\bullet'')$ is a universal family, so this implies that every α'' -stable vector bundle is $T(\alpha)$ -stable and viceversa, so α'' belongs to the same stability chamber as $T(\alpha)$ and $\Phi' = T$. \square

Lemma 4.6.23. *Suppose that $g \geq 4$. For every basic transformation $T \in \mathcal{T}$ such that $T \neq \text{Id}_{\mathcal{T}} = (\text{Id}, 1, \mathcal{O}_X, 0)$ and a generic α -stable parabolic vector bundle (E, E_\bullet) we have $T(E, E_\bullet) \not\cong (E, E_\bullet)$.*

Proof. Assume that $T \neq \text{Id}_{\mathcal{T}}$ but $T = (\sigma, s, L, H)$ acts as the identity on $\mathcal{M}(r, \alpha, \xi)$. First, let us prove that $H = 0$. Assume that $H \neq 0$. Let $x \in D$ such that $H \geq kx$, but $H \not\geq (k+1)x$. Then for every $(E, E_\bullet) \in \mathcal{M}(r, \alpha, \xi)$

$$(\sigma, s, L, 0) \circ \mathcal{H}_H(E, E_\bullet) \cong (E, E_\bullet)$$

By Lemma 4.1.11 and Lemma 4.1.12, for a generic $(E, E_\bullet) \in \mathcal{M}(r, \alpha, \xi)$ if E'_\bullet is the filtration obtained by changing the step $E_{x,k}$ on $x \in D$ to $E'_{x,k}$ for some

$$E_{x,k-1} \subsetneq E''_{x,k} \subsetneq E_{x,k+1}$$

then (E, E'_\bullet) is α -stable. Then there is a short exact sequence

$$0 \longrightarrow \mathcal{H}_H(E, E'_\bullet) \longrightarrow \mathcal{H}_{H-kx}(E, E'_\bullet) \longrightarrow E|_x/E'_{x,k} \longrightarrow 0$$

Therefore, as $E'_{x,k}$ changes through all possible steps in the filtration, then the underlying vector bundle of $\mathcal{H}_H(E, E'_\bullet)$ varies. Nevertheless, as

$$\mathcal{H}_H(E, E'_\bullet) = (\sigma, s, L, 0)^{-1}(E, E'_\bullet) \cong (\sigma^{-1}, s, \sigma^* L^{-s}, 0)(E, E_\bullet)$$

then the underlying vector bundle of $\mathcal{H}_H(E, E'_\bullet)$ must be isomorphic to E for every $E'_{x,k}$, so we obtain a contradiction and $H = 0$.

Similarly, σ must fix every parabolic point. Otherwise, if $\sigma(x) \neq x$ for some $x \in D$, then taking any variation $E'_{x,k}$ of the parabolic structure at x we would obtain that

$$(\sigma, s, L, 0)(E, E'_\bullet) \cong (E, E'_\bullet)$$

Nevertheless, the left hand side of the equation has constant parabolic structure over x , while the parabolic structure on the right hand side varies over x .

Now, let us prove that $s = 1$. If $s = -1$, for every parabolic vector bundle (E, E_\bullet) and every $x \in D$ the isomorphism $\sigma^*(L \otimes E)^\vee \cong E$ induce a nondegenerate symmetric map

$$\omega : E|_x \otimes E|_x \longrightarrow L^{-1}|_x$$

Under the isomorphism $T(E, E_\bullet) \cong E$ the k -th step of the parabolic structure $E_{x,k} \subset E|_x$ is sent to

$$E_{x,k}^\omega = \{v \in E|_x \mid \forall u \in E_{x,k} \omega(u, v) = 0\}$$

First, observe that this transformation inverts the filtration, i.e., $E_{x,k}$ is sent to $E_{x,r-k+1}$. We know that T preserves the parabolic structure, so $r = 2$. Nevertheless, as ω is nondegenerate, $E_{x,1}^\omega \cap E_{x,1} = 0$, so $E_{x,1} \neq E_{x,1}^\omega$ and T cannot preserve the parabolic structure.

Now, let $S \in \mathcal{T}$ be any basic transformation such that $S(\mathcal{O}_X) = \xi$. Then $S \circ T \circ S^{-1} \neq \text{Id}_{\mathcal{T}}$, but $S \circ T \circ S^{-1} : \mathcal{M}(r, S^{-1}(\alpha), \mathcal{O}_X) \longrightarrow \mathcal{M}(r, S^{-1}(\alpha), \mathcal{O}_X)$ is the identity on $\mathcal{M}(r, S^{-1}(\alpha), \mathcal{O}_X)$. Therefore, we can assume without loss of generality that $\xi \cong \mathcal{O}_X$. In this case, taking determinants yields

$$\mathcal{O}_X \cong \det(E) \cong \det(\sigma^* L \otimes E) \cong \sigma^* L^{-r}$$

Therefore, $L^r \cong \mathcal{O}_X$.

Then $T = (\sigma, 1, L, 0)$ preserves α and ξ for any system of weights and every line bundle. Moreover, by Corollary 4.1.4 for any system of weights α' , there exists an open subset $\mathcal{U} \subset \mathcal{M}(r, \alpha', \xi)$ whose complement has codimension at least 2 and such that all the parabolic vector bundles in \mathcal{U} are α stable. Consider the morphism $T : \mathcal{M}(r, \alpha', \xi) \longrightarrow \mathcal{M}(r, \alpha', \xi)$. Over \mathcal{U} this morphism is the identity, so $T = \text{Id}_{\mathcal{M}(r, \alpha', \xi)}$. Therefore, we can assume that α is any system of weights. For example, we can assume that it is concentrated.

Projectivizing, if $\sigma^*(L \times E) \cong E$ for any stable E then $\sigma^*(\mathbb{P}(E)) \cong \mathbb{P}(E)$ for all stable E . As σ acts faithfully on the moduli space of stable projective bundles we obtain that $\sigma = \text{Id}$. Finally let us prove that $L = \text{Id}$. Using Narasimhan-Seshadri Theorem [NS65] the space of stable vector bundles with trivial determinant is in bijection with the space of irreducible representations

$$\rho : \pi_1(X) \longrightarrow U(r)$$

modulo conjugation by $U(r)$. If ρ is a representation associated to E and $l : \pi_1(X) \longrightarrow \mathbb{C}^*$ is a representation associated to L then $l\rho : \pi_1(X) \longrightarrow U(r)$ is a representation associated to E . Two representations correspond to the same vector bundle if and only if one is obtained from the other by conjugation. Nevertheless, taking traces we obtain that

$$\text{tr}(l\rho) = l \text{tr}(\rho) \neq \text{tr}(\rho)$$

unless $\text{tr}(\rho) = 0$ or $l \cong 1$. Therefore $l\rho$ and ρ cannot be related by conjugation unless $\text{tr}(\rho) = 0$ or $l \cong 1$. Then either $L = \mathcal{O}_X$, or for a generic E we obtain that $L \otimes E \not\cong E$. □

Theorem 4.6.24. *Let (X, D) be a smooth projective curve of genus $g \geq 6$ and let α be a full flag generic system of weights over (X, D) of rank r . Let ξ be a line bundle over X . Then the automorphism group of $\mathcal{M}(r, \alpha, \xi)$ is the subgroup of \mathcal{T} consisting on basic transformations T such that*

- $T(\xi) \cong \xi$
- $T(\alpha)$ is in the same stability chamber as α

Proof. If we take $(X', D') = (X, D)$ and $\alpha' = \alpha$ in Theorem 4.6.22, we obtain that if $\Phi : \mathcal{M}(r, \alpha, \xi) \rightarrow \mathcal{M}(r, \alpha, \xi)$ is an automorphism then there must exist a basic transformation $T \in \mathcal{T}$ such that $\Phi(E, E_\bullet) \cong T(E, E_\bullet)$. Nevertheless, this implies that $\xi' \cong T(\xi)$ and $T(\alpha)$ is in the same stability chamber as α .

Clearly, the subset of transformations T preserving ξ and the chamber of α form a subgroup of \mathcal{T} . As the group structure of \mathcal{T} coincides with the composition of morphisms between moduli spaces of parabolic vector bundles, then this subgroup projects to the group of automorphisms of $\mathcal{M}(r, \alpha, \xi)$. To prove the theorem it is enough to check that if $T, T' \in \mathcal{T}$ are different elements in \mathcal{T} which satisfy the restrictions then the induced automorphisms $T, T' \in \text{Aut}(\mathcal{M}(r, \alpha, \xi))$ are different. Composing $T' \circ T^{-1} \in \text{Aut}(\mathcal{M}(r, \alpha, \xi))$, this is equivalent to prove that if $T \neq \text{Id}$ then there exists at least a parabolic vector bundle (E, E_\bullet) such that $T(E, E_\bullet) \neq (E, E_\bullet)$. Now we simply apply the previous Lemma. □

4.7 Birational geometry

In this section we will analyze the birational geometry of the moduli space of parabolic vector bundles with fixed determinant and, in particular, in the birational automorphisms of the moduli space. Boden and Yokogawa [BY99, Theorem 6.1] proved that for $g \geq 3$, if α is a full flag system of weights and ξ is any line bundle over (X, D) then $\mathcal{M}(r, \alpha, \xi)$ is a rational variety of dimension

$$\dim(\mathcal{M}(r, \alpha, \xi)) = (r^2 - 1)(g - 1) + |D| \frac{r^2 - r}{2} = m$$

Therefore, we know that for every (X, D) of genus g and $|D|$ parabolic points there is a birational map

$$\mathcal{M}(X, r, \alpha, \xi) \dashrightarrow \mathbb{P}^m$$

In particular

$$\text{Aut}_{\text{Bir}}(\mathcal{M}(X, r, \alpha, \xi)) = \text{Aut}_{\text{Bir}}(\mathbb{P}^m)$$

It is then clear that two moduli spaces $\mathcal{M}(X, r, \alpha, \xi)$ and $\mathcal{M}(X', r', \alpha', \xi')$ are birationally equivalent if and only if their dimension coincide.

In a first approach, this result closes the problem of understanding the rational geometry of the moduli space and blocks the possibility of a “birational Torelli”

type theorem. However, there is no control “a priori” of how far are the birational equivalences that relate two moduli spaces $\mathcal{M}(X, r, \alpha, \xi)$ and $\mathcal{M}(X', r', \alpha', \xi')$ from extending to an isomorphism. More precisely, we know that if these moduli spaces have the same dimension, then there exist open subsets $\mathcal{U} \subset \mathcal{M}(X, r, \alpha, \xi)$ and $\mathcal{U}' \subset \mathcal{M}(X', r', \alpha', \xi')$ and an isomorphism $\Phi : \mathcal{U} \xrightarrow{\sim} \mathcal{U}'$. Nevertheless, \mathcal{U} and \mathcal{U}' can be “small” open subsets in the sense that their complement can have codimension 1 (and in fact, they are expected to do so). In this section, we will be interested in understanding the birational geometry of the moduli spaces when we restrict the allowed rational maps to those that can be extended to subsets whose complement has codimension at least 3.

We will start by generalizing some of the core lemmata in section 4.3 so they can work in the k -birational setting.

Definition 4.7.1. *Let \mathcal{X} and \mathcal{X}' be two varieties. We say that \mathcal{X} and \mathcal{X}' are k -birational if there exist open subsets $\mathcal{U} \subset \mathcal{X}$ and $\mathcal{U}' \subset \mathcal{X}'$ and an isomorphism $\Phi : \mathcal{U} \xrightarrow{\sim} \mathcal{U}'$ such that*

$$\begin{aligned} \text{codim}(\mathcal{X} \setminus \mathcal{U}) &\geq k \\ \text{codim}(\mathcal{X}' \setminus \mathcal{U}') &\geq k \end{aligned}$$

In particular, \mathcal{X} and \mathcal{X}' are birational if they are at least 1-birational. Given a variety \mathcal{X} , we denote by $\text{Aut}_{k\text{-Bir}}(\mathcal{X})$ the space of k -birational automorphisms of \mathcal{X} .

The study of k -birational maps instead of rational maps is useful in many contexts. For example, some geometric invariants like the Picard group are invariant under 2-birational maps, but not under 1-birational ones. Hartog’s theorem proves that if \mathcal{X} and \mathcal{X}' are 2-birationally equivalent normal algebraic varieties then $\Gamma(\mathcal{X}) \cong \Gamma(\mathcal{X}')$. In the context of the moduli space of vector bundles (and parabolic vector bundles), we know that for $g \geq 4$ the moduli space of (parabolic) Higgs bundles is 3-birationally equivalent to the cotangent bundle of the moduli space of (parabolic) vector bundles. The fact that they are 3-birational and not just 2-birational was used in Section 4.3 in order to control the geometry of some special fibers of the Hitchin map.

As we cannot distinguish the moduli spaces nor the isomorphisms between them at the 1-birational level, we will focus on the k -birational maps between moduli spaces for $k > 1$ and prove that if we restrict to 3-birational maps we obtain enough information to be able to describe a birational Torelli type theorem and obtain an analogue of Theorem 4.6.22 which categorizes all the 3-birational maps. Although we believe that the presented results will remain true for 2-birational maps as well and that the classification could be attempted with similar techniques as the ones presented in this work, due to some technical requisites in the proof exposed in this article we will restrict ourselves to the analysis 3-birational maps.

Corollary 4.7.2. *Suppose that $g \geq 4$. Let $\mathcal{V} \subset \mathcal{M}(r, \alpha, \xi)$ be an open subset whose complement has codimension at least 3. Then the complement of $T^*\mathcal{V} \cap H^{-1}(\mathcal{D}_U)$ inside $H^{-1}(\mathcal{D}_U)$ has codimension at least 2.*

Proof. Let $\mathcal{Z} = \mathcal{M}(r, \alpha, \xi) \setminus \mathcal{V}$ and let $m = \dim(\mathcal{M}(r, \alpha, \xi))$. As $g \geq 4$, by Proposition 4.2.7 we know that

$$\dim(\mathcal{M}_{K(D)}(r, \alpha, \xi) \setminus T^*\mathcal{M}(r, \alpha, \xi)) \leq 2m - 3$$

Therefore, if we denote $\mathcal{E} = \mathcal{M}_{K(D)}(r, \alpha, \xi) \setminus T^*\mathcal{M}(r, \alpha, \xi)$ then

$$\dim(E \cap H^{-1}(\mathcal{D}_U)) \leq 2m - 3$$

Let us prove that $\dim(T^*\mathcal{Z} \cap H^{-1}(\mathcal{D}_U)) \leq 2m - 3$. In that case we would have

$$\begin{aligned} \dim(H^{-1}(\mathcal{D}_U) \setminus (T^*\mathcal{V} \cap H^{-1}(\mathcal{D}_U))) &= \dim((\mathcal{E} \cap H^{-1}(\mathcal{D}_U)) \cup (T^*\mathcal{Z} \cap H^{-1}(\mathcal{D}_U))) \\ &\leq 2m - 3 = \dim(H^{-1}(\mathcal{D}_U)) - 2 \end{aligned}$$

First, assume that $\dim(\mathcal{Z}) \leq m - 3$. Then $\dim(T^*\mathcal{Z}) \leq 2m - 3$, so $\dim(T^*\mathcal{Z} \cap H^{-1}(\mathcal{D}_U)) \leq 2m - 3$. \square

Lemma 4.7.3. *Let $g \geq 4$ and let $\mathcal{V} \subset \mathcal{M}(r, \alpha, \xi)$ be any open subset whose complement has codimension at least 3. Let $\mathcal{R}_{\mathcal{V}} \subset T^*\mathcal{V}$ be the union of the complete rational curves in $T^*\mathcal{V}$. Then \mathcal{D} is the closure of $H(\mathcal{R}_{\mathcal{V}})$ in W .*

Proof. The proof is analogous of Lemma 4.3.3. Let $H_{\mathcal{V}} : T^*\mathcal{V} \rightarrow W$ be the restriction of the Hitchin map H to $T^*\mathcal{V}$. If $\mathbb{P}^1 \hookrightarrow T^*\mathcal{V}$ is a complete rational curve, then it must be contained in a fiber of the Hitchin map. If $s \in W \setminus D$, then $H^{-1}(s)$ is an abelian variety, so $H_{\mathcal{V}}^{-1}(s)$ is an open subset of an abelian variety and, therefore, it does not admit any nonconstant morphism from \mathbb{P}^1 . Therefore, we only have to prove that for a generic s in every irreducible component of \mathcal{D} the fiber $H_{\mathcal{V}}^{-1}(s)$ contains a complete rational curve. In this case $H_{\mathcal{V}}(\mathcal{R}_{\mathcal{V}})$ is dense in \mathcal{D} and the lemma holds.

For the components \mathcal{D}_x for $x \in D$, we can proceed just as in the proof of Proposition 4.3.2, changing the subset $\mathcal{U} \subset \mathcal{M}(r, \alpha, \xi)$ parameterizing $(1, 0)$ -stable parabolic vector bundles (E, E_{\bullet}) such that $H^0(\text{PEnd}_0(E, E_{\bullet})(x)) = 0$ with the following open nonempty subset \mathcal{U}' . For every $(E, E_{\bullet}) \in \mathcal{Z} = \mathcal{M} \setminus \mathcal{V}$ and every $1 \leq k < r$, let us consider the family of quasi-parabolic vector bundles over \mathbb{P}^1 obtained by changing the k -th step of the filtration of $E|_x$ to all admissible subspaces $E'_{x,k}$ such that

$$E_{x,k+1} \subsetneq E'_{x,k} \subsetneq E_{x,k-1}$$

Consider the union of all the α -stable points (E, E'_{\bullet}) in such families. As the codimension of \mathcal{Z} in $\mathcal{M}(r, \alpha, \xi)$ is at least 3 and the families are at most 1-dimensional, then union of all the families must have positive codimension and therefore, there exists some open nonempty subset $\mathcal{W} \subset \mathcal{M}(r, \alpha, \xi)$ whose points are not in the image of any family. Now take $\mathcal{U}' = \mathcal{U} \cap \mathcal{V} \cap \mathcal{W}$ and repeat the argument in 4.3.2.

For a generic $x \in \mathcal{D}_U$, X_s has a unique singularity which is a node not lying over a parabolic point. Then $H^{-1}(s)$ is an uniruled variety of dimension m . Let $Z = (\mathcal{M}_{K(D)}(r, \alpha, \xi) \setminus T^*\mathcal{V}) \cap H^{-1}(\mathcal{D}_U)$. If $g \geq 4$, by Corollary 4.7.2 the codimension of Z in $H^{-1}(\mathcal{D}_U)$ is at least 2. Let $S = H(Z)$. If $\dim(S) < m - 1$ then for every $s \in \mathcal{D}_U \setminus S$, $H^{-1}(s) = H_{\mathcal{V}}^{-1}(s)$, so the fiber of the (restricted) Hitchin map contains a complete rational curve.

On the other hand, if $\dim(S) = m - 1$, then $H|_Z : Z \rightarrow \mathcal{D}_U$ is dominant and, therefore, the generic fiber has dimension $\dim(Z) - \dim(\mathcal{D}_U) \leq m - 2$. Then, for a generic $s \in \mathcal{D}_U$, $Z \cap H^{-1}(s)$ has codimension at least 2 in $H^{-1}(s)$. Therefore $H^{-1}(s) \setminus H_{\mathcal{V}}^{-1}(s)$ has codimension at least 2 in $H^{-1}(s)$ and $H_{\mathcal{V}}^{-1}(s)$ must contain a complete rational curve. \square

Proposition 4.7.4. *Let $\mathcal{V} \subset \mathcal{M}(r, \alpha, \xi)$ be an open subset whose complement has codimension at least 2. Then the global algebraic functions $\Gamma(T^*\mathcal{V})$ produce a map*

$$\tilde{h} : T^*\mathcal{V} \longrightarrow \text{Spec}(\Gamma(T^*\mathcal{V})) \cong W \cong \mathbb{C}^m$$

which is the restriction of the parabolic Hitchin map to $T^\mathcal{V}$ up to an isomorphism of \mathbb{C}^m , where $m = \dim W$. Moreover, consider the action of \mathbb{C}^* on $T^*\mathcal{V}$ given by dilatation on the fibers. Then there is a unique \mathbb{C}^* action on W such that \tilde{h} is \mathbb{C}^* -equivariant*

Proof. For $\mathcal{V} = \mathcal{M}(r, \alpha, \xi)$ this was proved in Proposition 4.3.4. As $T^*\mathcal{V} \subset T^*\mathcal{M}(r, \alpha, \xi)$ is an open subset whose complement has codimension at least 2 and $T^*\mathcal{M}(r, \alpha, \xi)$ is smooth then by Hartog's theorem we know that $\Gamma(T^*\mathcal{V}) = \Gamma(T^*\mathcal{M}(r, \alpha, \xi))$ and the Proposition follows. \square

Theorem 4.7.5. *Let (X, D) and (X', D') be two smooth projective curves of genus $g \geq 4$ and $g' \geq 4$ respectively with set of marked points $D \subset X$ and $D' \subset X'$. Let ξ and ξ' be line bundles over X and X' respectively, and let α and α' be full flag generic systems of weights over (X, D) and (X', D') respectively. Then if $\mathcal{M}(X, r, \alpha, \xi)$ is 3-birational to $\mathcal{M}(X', r', \alpha', \xi')$ then $r = r'$ and (X, D) is isomorphic to (X', D') , i.e., there exists an isomorphism $X \cong X'$ sending the set D to D' .*

Proof. The proof will be completely analogous to the one given for Theorem 4.3.6. Let $\mathcal{V} \subset \mathcal{M}(X, r, \alpha, \xi)$ and $\mathcal{V}' \subset \mathcal{M}(X', r', \alpha', \xi')$ be open subsets whose complement has codimension 3 and let $\Phi : \mathcal{V} \rightarrow \mathcal{V}'$ be the 3-birational morphism between both moduli spaces. In particular

$$\begin{aligned} (r-1) \left[(r-1)(g-1) + \frac{|D|}{2}r \right] &= \dim(\mathcal{V}) \\ &= \dim(\mathcal{V}') = (r'-1) \left[(r'-1)(g'-1) + \frac{|D'|}{2}r' \right] \end{aligned} \quad (4.7.1)$$

On the other hand, by Proposition 4.7.4 there must exist an algebraic \mathbb{C}^* -equivariant isomorphism $f : W \cong \text{Spec}(\Gamma(T^*\mathcal{V})) \xrightarrow{\sim} \text{Spec}(\Gamma(T^*\mathcal{V}')) \cong W'$ such that the following diagram commutes

$$\begin{array}{ccc} T^*\mathcal{V} & \xrightarrow{d(\Phi^{-1})} & T^*\mathcal{V}' \\ \tilde{h} \downarrow & & \downarrow \tilde{h} \\ W & \xrightarrow{f} & W' \end{array}$$

As f is \mathbb{C}^* -equivariant, it must preserve the filtration by subspaces in terms of the decay and it must send the subspace of maximum decay $|\lambda|^r$ of W to the subspace of maximum decay $|\lambda|^{r'}$ of W' . Therefore, the number of steps of the filtration must be the same and the spaces of top decay must have the same dimension. As the filtrations of W and W' have $r-1$ and $r'-1$ steps respectively, then $r = r'$. The dimension of such subspaces are the dimensions of $W_r = H^0(K_X^r D^{r-1})$ and $W'_r = H^0(K_{X'}^r (D')^{r-1})$ respectively, so

$$(r-1)(2g-2+|D|) = h^0(K_X^r D^{r-1}) = h^0(K_{X'}^r (D')^{r-1}) = (r-1)(2g'-2+|D'|)$$

This, together with equation (4.7.1) proves that $g = g'$ and $|D| = |D'|$. As $d(\Phi^{-1})$ is an isomorphism, it maps complete rational curves in $T^*\mathcal{V}$ to complete rational curves in $T^*\mathcal{V}'$. By Lemma 4.7.3, f sends the locus of singular spectral curves $\mathcal{D} \subset W$ to the locus of singular spectral curves $\mathcal{D}' \subset W'$. Moreover, we know that $f(W_r) = W'_r$, so if we let $\mathcal{C} = \mathcal{D} \cap W_r$ and $\mathcal{C}' = \mathcal{D}' \cap W'_r$ we obtain that $f(\mathcal{C}) = \mathcal{C}'$.

By Proposition 4.3.5, the dual variety of $\mathbb{P}(\mathcal{C}_X)$ in $\mathbb{P}(W_r)$ is $X \subset \mathbb{P}(W_r^*)$ and, similarly, the dual variety of $\mathbb{P}(\mathcal{C}'_X)$ in $\mathbb{P}(W'_r)$ is $X' \subset \mathbb{P}((W'_r)^*)$, so f induces an isomorphism $f^\vee : \mathbb{P}(W_r^*) \rightarrow \mathbb{P}((W'_r)^*)$ that sends X to X' . Moreover, the dual of the rest of the components $\mathbb{P}(\mathcal{C}_x)$ of $\mathbb{P}(\mathcal{C})$ correspond to the divisor $D \subset X \subset \mathbb{P}(W_r^*)$ and the dual of the components $\mathbb{P}(\mathcal{C}'_x)$ of $\mathbb{P}(\mathcal{C}')$ correspond to the divisor $D' \subset X' \subset \mathbb{P}((W'_r)^*)$, so f^\vee must send D to D' . Therefore, f^\vee induces an isomorphism $f^\vee : (X, D) \xrightarrow{\sim} (X', D')$. \square

In contrast with the usual Torelli theorem, where there exist several non-isomorphic moduli spaces of parabolic vector bundles for the same curve (X, D) depending on the stability and topological data, in the case of k -birational geometry we can state a hard reciprocal of the Torelli theorem

Proposition 4.7.6. *Let (X, D) be a marked smooth projective curve of genus $g \geq 1 + \frac{k-1}{r-1}$. Let ξ and ξ' be line bundles over X and let α and α' be full flag generic systems of weights of rank r over (X, D) . Then there is a k -birational map*

$$\mathcal{M}(r, \alpha, \xi) \dashrightarrow \mathcal{M}(r, \alpha', \xi')$$

In particular, if $g \geq 3$, $\mathcal{M}(r, \alpha, \xi)$ and $\mathcal{M}(r, \alpha', \xi')$ are 3-birational.

Proof. Let $d = \deg(\xi)$ and $d' = \deg(\xi')$. Let us write $d' - d = rm - k$ for some $0 \leq k < r$. Let $x \in D$ be any parabolic point. Then

$$\deg(\mathcal{T}_{\mathcal{O}_X(mx)} \circ \mathcal{H}_{kx}(\xi)) = \deg(\xi')$$

Therefore, there exists a line bundle L of degree zero such that

$$\xi' = L^r \otimes (\mathcal{T}_{\mathcal{O}_X(mx)} \circ \mathcal{H}_{kx}(\xi)) = \mathcal{T}_{L(mx)} \circ \mathcal{H}_{kx}(\xi)$$

Take $T = (\text{Id}, 1, L(mx), kx)$. Then T induces an isomorphism

$$T : \mathcal{M}(r, \alpha, \xi) \longrightarrow \mathcal{M}(r, T(\alpha), \xi')$$

By Corollary 4.1.4 there exists an open subset $\mathcal{U} \subset \mathcal{M}(r, T(\alpha), \xi')$ whose complement has codimension at least 3 parameterizing α' -stable parabolic vector bundles in $\mathcal{M}(r, T(\alpha), \xi')$. Similarly, there exists $\mathcal{U}' \subset \mathcal{M}(r, \alpha', \xi')$ whose complement have codimension at least 3 parameterizing $T(\alpha)$ -stable parabolic vector bundles in $\mathcal{M}(r, \alpha', \xi')$. Then \mathcal{U} and \mathcal{U}' can be canonically identified as the moduli space of parabolic vector bundles of rank r and determinant ξ which are both $T(\alpha)$ -stable and α' -stable. Finally, $T^{-1}(\mathcal{U}) \subset \mathcal{M}(r, \alpha, \xi)$ is an open subset whose complement has codimension at least k and we have an isomorphism $T^{-1}(\mathcal{U}) \cong \mathcal{U}'$ so the moduli spaces are 3-birational. \square

Observe that we obtain analogues of this Proposition in the k -birational category by just increasing the genus condition, while the Torelli theorem holds in the k -birational category for any $g \geq 4$.

Now let $\Phi : \mathcal{M}(X, r, \alpha, \xi) \dashrightarrow \mathcal{M}(X', r', \alpha', \xi')$ be a 3-birational isomorphism. By the 3-birational version of the Torelli Theorem we have $r = r'$ and the 3-birational map Φ induces an isomorphism $\sigma : X \rightarrow X'$ which sends the set D to D' . Pulling back by σ , we obtain a 3-birational map

$$\Phi' = \Sigma_\sigma \circ \Phi : \mathcal{M}(X, r, \alpha, \xi) \dashrightarrow \mathcal{M}(X, r, \sigma^* \alpha, \sigma^* \xi')$$

Let $\alpha'' = \sigma^* \alpha$ and $\xi'' = \sigma^* \xi'$. Let $\mathcal{V} \subset \mathcal{M}(X, r, \alpha, \xi)$ and $\mathcal{V}'' \subset \mathcal{M}(X, r, \alpha'', \xi'')$ be open subsets whose respective complements have codimension at least 3 such that $\Phi' : \mathcal{V} \rightarrow \mathcal{V}''$ is an isomorphism. Then the differential induces an isomorphism $d(\Phi'^{-1}) : T^* \mathcal{V} \rightarrow T^* \mathcal{V}''$. Let $h : T^* \mathcal{V} \rightarrow W$ and $h'' : T^* \mathcal{V}'' \rightarrow W$ denote the restriction of the Hitchin morphism to \mathcal{V} and \mathcal{V}'' respectively. Since both moduli spaces are built over the same marked curve (X, D) and with the same rank r , the Hitchin space is the same. By Proposition 4.7.4, there exists a \mathbb{C}^* -equivariant automorphism $f : W \rightarrow W$ such that the following diagram commutes

$$\begin{array}{ccc} T^* \mathcal{V} & \xrightarrow{d(\Phi'^{-1})} & T^* \mathcal{V}'' \\ h \downarrow & & \downarrow h'' \\ W & \xrightarrow{f} & W \end{array}$$

By Lemma 4.7.3, $f : W \rightarrow W$ must preserve the discriminant locus, i.e., $f(\mathcal{D}) = \mathcal{D}$. We know that it is \mathbb{C}^* -equivariant, so using Lemma 4.6.11, f preserves the decomposition $W = \bigoplus_{k \geq 1} W_k$ and its restrictions $f_k : W_k \rightarrow W_k$ are linear. For each $k \geq 1$, let $h_k : T^* \mathcal{V} \rightarrow W_k$ and $h''_k : T^* \mathcal{V}'' \rightarrow W_k$ denote the compositions of h and h'' with the projection $W \rightarrow W_k$ respectively. In particular, for each $k \geq 1$ the following diagram commutes

$$\begin{array}{ccc} T^* \mathcal{V} & \xrightarrow{d(\Phi'^{-1})} & T^* \mathcal{V}'' \\ h_k \downarrow & & \downarrow h''_k \\ W_k & \xrightarrow{f_k} & W_k \end{array}$$

Lemma 4.7.7. *Let $g \geq 4$. Let $f_r : W_r \rightarrow W_r$ be the \mathbb{C}^* -equivariant map on the Hitchin space such that the following diagram commutes*

$$\begin{array}{ccc} T^* \mathcal{V} & \xrightarrow{d(\Phi'^{-1})} & T^* \mathcal{V}'' \\ h_r \downarrow & & \downarrow h'_r \\ W_r & \xrightarrow{f_r} & W_r \end{array} \tag{4.7.2}$$

Then for every $k \geq 0$ and every $x_0 \in X$

$$f_r(H^0(K^r D^{r-1}(-kx_0))) = H^0(K^r D^{r-1}(-kx_0))$$

Proof. As $d(\Phi^{-1})$ is an isomorphism, it maps complete rational curves on $T^*\mathcal{V}$ to complete rational curves on $T^*\mathcal{V}''$. By Lemma 4.7.3, the morphism f must preserve $\mathcal{C} = \mathcal{D} \cap W_r$. Therefore, the associated map of dual varieties is an automorphism of the marked curve (X, D) . Through the previous discussion, we proved that we can assume that the induced automorphism of the curve X is the identity, so we can just proceed as in the proof of Lemma 4.6.13. \square

Once we have proven the 3-birational version of the Torelli theorem and the previous Lemma, we automatically obtain that if $(E, E_\bullet) \in \mathcal{V}$ is a generic parabolic vector bundle and $\Phi(E, E_\bullet) = (E'', E''_\bullet) \in \mathcal{V}''$ is also generic in the sense of Lemma 4.1.7 then Lemmas 4.6.14, 4.6.15 and 4.6.16 hold and we obtain that there is an isomorphism

$$\Phi_{\text{SPEnd}_0} : \text{PEnd}_0(E, E_\bullet) \cong \text{PEnd}_0(E'', E''_\bullet)$$

Moreover, we obtain an analogue of Lemma 4.6.18

Lemma 4.7.8. *Suppose that $g \geq 4$. For each $x \in X$, and every $k > 1$, the linear subspace*

$$H^0(K^k D^{k-1}(-x)) \subseteq W_k$$

is preserved by the linear map $f_k : W_k \rightarrow W_k$.

Proof. Let $\tilde{\mathcal{V}} \subset \mathcal{V}$ the open subset of parabolic vector bundles $(E, E_\bullet) \in \mathcal{V}$ such that both (E, E_\bullet) and $\Phi(E, E_\bullet)$ are generic in the sense of Lemma 4.1.7. Applying Corollary 4.2.3 to $L = K(D - x)$ and the open subsets $\tilde{\mathcal{V}}$ and $\tilde{\mathcal{V}}'' = \Phi(\tilde{\mathcal{V}})$ we obtain that the linear subspace

$$\bigoplus_{k=2}^r H^0(K^k D^{k-1}(-kx)) \subseteq W$$

is the space generated by the images $h(H^0(\text{SPEnd}_0(E, E_\bullet) \otimes K(D - x)))$ both when (E, E_\bullet) runs over $\tilde{\mathcal{V}}$ and when (E, E_\bullet) runs over $\tilde{\mathcal{V}}''$.

By Lemma 4.6.15, for every $(E, E_\bullet) \in \tilde{\mathcal{V}}$, if $(E'', E''_\bullet) = \Phi(E, E_\bullet) \in \tilde{\mathcal{V}}''$, then the image of $H^0(\text{SPEnd}_0(E, E_\bullet) \otimes K(D - x))$ by $d(\Phi^{-1})$ is $H^0(\text{SPEnd}_0(E'', E''_\bullet) \otimes K(D - x))$. Therefore f preserves $\bigoplus_{k=2}^r H^0(K^k D^{k-1}(-kx))$. As it is diagonal, f_k preserves $H^0(K^k D^{k-1}(-kx))$.

For $k > 1$ the curve X is embedded in $\mathbb{P}(W_k^*)$ through the linear system $|K^r D^{r-1}|$. The spaces $H^0(K^k D^{k-1}(-kx))$ for $x \in X$ correspond to the osculating k -spaces of X at x , $\text{Osc}_k(x)$. As $\mathbb{P}(f_k)$ preserves $\text{Osc}_k(x)$, by Lemma 4.6.19 it preserves $\text{Osc}_1(x)$ and, therefore, f_k must preserve the hyperplanes

$$H^0(K^k D^{k-1}(-x)) \subset H^0(K^k D^{k-1})$$

for every $x \in X$. \square

From this result we obtain the following Lemma, whose proof is exactly the same as Lemma 4.6.21

Lemma 4.7.9. *Suppose that $g \geq 6$. Let $(E, E_\bullet) \in \tilde{\mathcal{V}} \subset \mathcal{V}$ and let $(E'', E''_\bullet) = \Phi(E, E_\bullet)$. Then $\text{PEnd}_0(E, E_\bullet)$ and $\text{PEnd}_0(E'', E''_\bullet)$ are isomorphic as Lie algebra bundles over X .*

Now we are ready to generalize Theorem 4.6.22 to the 3-birational setting.

Theorem 4.7.10. *Let (X, D) and (X', D') be two smooth projective curves of genus $g \geq 6$ and $g' \geq 6$ respectively with a set of marked points $D \subset X$ and $D' \subset X'$. Let ξ and ξ' be line bundles over X and X' respectively, and let α and α' be full flag generic systems of weights over (X, D) and (X', D') respectively. Let*

$$\Phi : \mathcal{M}(X, r, \alpha, \xi) \dashrightarrow \mathcal{M}(X', r', \alpha', \xi')$$

be a 3-birational map. Then

1. $r = r'$
2. (X, D) is isomorphic to (X', D') , i.e., there exists an isomorphism $\sigma : X \xrightarrow{\sim} X'$ sending D to D' .
3. There exists a basic transformation T such that
 - $\sigma^* \xi' \cong T(\xi)$
 - For every $(E, E_\bullet) \in \mathcal{M}(r, \alpha, \xi)$, $\sigma^* \Phi(E, E_\bullet) \cong T(E, E_\bullet)$

Proof. By the 3-birational version of the Torelli Theorem (Theorem 4.7.5) we have $r = r'$ and the 3-birational map Φ induces an isomorphism $\sigma : X \rightarrow X'$ which sends the set D to D' . Pulling back by σ , we obtain a 3-birational map

$$\Phi' : \mathcal{M}(X, r, \alpha, \xi) \dashrightarrow \mathcal{M}(X, r, \sigma^* \alpha, \sigma^* \xi')$$

Let $\alpha'' = \sigma^* \alpha$ and $\xi'' = \sigma^* \xi'$. Let $\mathcal{V} \subset \mathcal{M}(X, r, \alpha, \xi)$ and $\mathcal{V}'' \subset \mathcal{M}(X, r, \alpha'', \xi'')$ be open subsets whose respective complements have codimension at least 3 such that $\Phi' : \mathcal{V} \rightarrow \mathcal{V}''$ is an isomorphism. Let $\tilde{\mathcal{V}} \subset \mathcal{V}$ be the subset of parabolic vector bundles $(E, E_\bullet) \in \mathcal{V}$ such that both (E, E_\bullet) and $(E'', E''_\bullet) = \Phi'(E, E_\bullet)$ are generic in the sense of Lemma 4.1.7. Then by Lemma 4.7.9 for every $(E, E_\bullet) \in \tilde{\mathcal{V}}$ we have that $\text{PEnd}_0(E, E_\bullet)$ and $\text{PEnd}_0(E'', E''_\bullet)$ are isomorphic as Lie algebra bundles over X . Then by Lemma 4.5.14 there exists a basic transformation $T = (\text{Id}, s, L, H)$ such that $(E'', E''_\bullet) \cong T(E, E_\bullet)$.

Up to this point we have proved that for every $(E, E_\bullet) \in \tilde{\mathcal{V}}$ there exists a basic transformation T such that $\Phi'(E, E_\bullet) = T(E, E_\bullet)$. Repeating the argument given in the proof of Theorem 4.6.22 we obtain that there exists some $T \in \mathcal{T}_{\xi, \xi''}$ such that for every $(E, E_\bullet) \in \tilde{\mathcal{V}}$, $\Phi'(E, E_\bullet) = T(E, E_\bullet)$. Repeating the argument in Theorem 4.6.22, let $\mathcal{W} \subset \mathcal{V}$ be the open subset consisting on parabolic vector bundles (E, E_\bullet) which are both α -stable and $T^{-1}(\alpha'')$ -stable. By Corollary 4.1.4, the complement of \mathcal{W} has codimension at least 2 in $\mathcal{M}(r, \alpha, \xi)$ and, in particular, $\mathcal{W} \cap \tilde{\mathcal{V}}$ is dense in \mathcal{W} . Therefore, for every map $\psi : \mathcal{W} \cap \tilde{\mathcal{V}} \rightarrow \mathcal{M}(r, \alpha'', \xi'')$ there exist at most a unique extension to \mathcal{W} . By construction of \mathcal{W} , we know that T gives a well defined map $T : \mathcal{W} \rightarrow \mathcal{M}(r, \alpha'', \xi'')$. Moreover, we know that $\Phi'|_{\mathcal{W} \cap \tilde{\mathcal{V}}} = T|_{\mathcal{W} \cap \tilde{\mathcal{V}}}$ and $\Phi'|_{\mathcal{W}}$ is another extension to \mathcal{W} , so $\Phi'|_{\mathcal{W}} = T|_{\mathcal{W}}$. Finally, let us prove that Φ' coincides with T over \mathcal{V} , i.e., that for every $(E, E_\bullet) \in \mathcal{M}(r, \alpha, \xi)$ such that Φ' is defined, $\Phi'(E, E_\bullet) = T(E, E_\bullet)$.

As α'' is a full flag system of weights, $\mathcal{M}(r, \alpha'', \xi'')$ is a fine moduli space for every ξ'' . Therefore, Φ' is represented by a parabolic vector bundle $(\mathcal{E}'', \mathcal{E}''_\bullet)$ over

$\mathcal{V} \times X$ whose fibers are α'' -stable as parabolic vector bundles over X . We have the following commutative diagram

$$\begin{array}{ccc}
 \Phi' \vdash & \xrightarrow{\hspace{10em}} & (\mathcal{E}'', \mathcal{E}_\bullet'') \\
 \downarrow T & & \downarrow \\
 \text{Hom}(\mathcal{V}, \mathcal{M}(r, \alpha'', \xi'')) & \xrightarrow{\sim} & \mathcal{M}(r, \alpha'', \xi'')(\mathcal{V}) \\
 \downarrow i^\# & & \downarrow i^* \\
 \text{Hom}(\mathcal{W}, \mathcal{M}(r, \alpha'', \xi'')) & \xrightarrow{\sim} & \mathcal{M}(r, \alpha'', \xi'')(\mathcal{W}) \\
 \downarrow T & & \downarrow \\
 T \vdash & \xrightarrow{\hspace{10em}} & T(\mathcal{E}, \mathcal{E}_\bullet)|_{\tilde{\mathcal{V}}}
 \end{array}$$

where $(\mathcal{E}, \mathcal{E}_\bullet)$ is the universal family of the moduli space $\mathcal{M}(r, \alpha, \xi)$. Therefore, $(\mathcal{E}'', \mathcal{E}_\bullet'')$ is an extension of $T(\mathcal{E}, \mathcal{E}_\bullet)|_{\mathcal{W}}$ from \mathcal{W} to \mathcal{V} . Clearly $T(\mathcal{E}, \mathcal{E}_\bullet)|_{\mathcal{V}}$ is a possible extension as a family of quasi-parabolic vector bundles over \mathcal{V} and the complement of \mathcal{W} in \mathcal{V} has codimension at least 2, so by Lemma 4.1.8 we have $(\mathcal{E}'', \mathcal{E}_\bullet'') \cong T(\mathcal{E}, \mathcal{E}_\bullet)|_{\mathcal{V}}$. Taking this isomorphism of families fiberwise we obtain the desired result. \square

Corollary 4.7.11. *Let (X, D) be a smooth projective curve of genus $g \geq 6$ and let α be a full flag generic system of weights over (X, D) of rank r . Let ξ be a line bundle over X . Then*

$$\text{Aut}_{3\text{-Bir}}(\mathcal{M}(r, \alpha, \xi)) = \mathcal{T}_\xi = \{T \in \mathcal{T} | T(\xi) \cong \xi\} < \mathcal{T}$$

Proof. Every basic transformation $T \in \mathcal{T}_\xi$ induce an isomorphism

$$T : \mathcal{M}(r, \alpha, \xi) \longrightarrow \mathcal{M}(r, T(\alpha), \xi)$$

By Corollary 4.1.4, there exist open subsets $\mathcal{U} \subset \mathcal{M}(r, \alpha, \xi)$ and $\mathcal{U}' \subset \mathcal{M}(r, T(\alpha), \xi)$ whose complement has codimension at least 3 parameterizing parabolic vector bundles of rank r and determinant ξ which are both α -stable and $T(\alpha)$ -stable. Therefore, there is an isomorphism $\Psi : \mathcal{U} \xrightarrow{\sim} \mathcal{U}'$. Composing with T , we obtain an isomorphism

$$\Psi^{-1} \circ T : T^{-1}(\mathcal{U}') \xrightarrow{\sim} \mathcal{U}$$

so we obtain a 3-birational map $\mathcal{M}(r, \alpha, \xi) \dashrightarrow \mathcal{M}(r, \alpha, \xi)$.

By the previous Theorem, every 3-birational automorphism is equivalent to one of the previous ones, so $\text{Aut}_{3\text{-Bir}}(\mathcal{M}(r, \alpha, \xi))$ is a quotient of \mathcal{T}_ξ . From Lemma 4.6.23, different basic transformations $T, T' \in \mathcal{T}_\xi$ induce different 3-birational automorphisms of the moduli space, so we obtain the desired equality. \square

4.8 Concentrated stability chamber

In the analysis of isomorphisms and k -birational transformations between moduli spaces of parabolic vector bundles held through the previous sections the systems of weights were allowed to belong to different stability chambers. This flexibility allowed us to describe transformations that transcended the limits of a stability chamber and relate moduli spaces for different choices of the stability and topological data.

Nevertheless, by Theorem 4.6.22 the possible basic transformations $T \in \mathcal{T}$ giving rise to automorphisms of a moduli space $\mathcal{M}(r, \alpha, \xi)$ must satisfy two compatibility conditions.

- $T(\xi) \cong \xi$
- $T(\alpha)$ belongs to the same stability chamber as α

While the first condition is easily computable and relies just on topological concerns, the second one depends on an analysis of the stability chamber where the system of weights α belongs. Therefore, it is possible that depending on the chamber certain basic generators of \mathcal{T} which preserve the determinant fail to preserve the stability and, therefore, they do not induce an automorphism.

Observe that if $T \in \mathcal{T}_\xi < \mathcal{T}$ then by Corollary 4.7.11 T induces a 3-birational transformation, but T induces an automorphism if and only if $T(\alpha)$ and α share the same stability chamber. Therefore, analyzing the stability chamber of $T(\alpha)$ for each $T \in \mathcal{T}_\xi$ is the same as studying the set of 3-birational automorphisms that extend to a regular automorphism of the whole moduli space.

For a general α an explicit analysis may depend greatly on the geometry of the curve, as the geometrical barriers in the space of systems of weights may vary with X in low genus. We seek for classification results that do not depend on the choice of the Riemann surface, we will work on two directions. On one hand, we will build invariants that allow us to distinguish stability chambers in a precise way for high genus. This will be done in Section 4.9. On the other hand, we will focus on studying some chamber where we can compute the stability conditions explicitly in low genus. In particular, in this section we will classify the automorphisms of the moduli space for a concentrated system of weights α .

The chamber of concentrated weights is of particular interest, as its interior corresponds to generic weights for which parabolic stability is roughly equivalent to the stability of the underlying vector bundle in the following sense (see, for example, [AG18b])

Lemma 4.8.1. *Let α be a generic concentrated system of weights. Let (E, E_\bullet) be a parabolic vector bundle. Then*

1. *If E is stable as a vector bundle then (E, E_\bullet) is α -stable as a parabolic vector bundle*
2. *(E, E_\bullet) is α -stable if and only if it is α -semistable*
3. *If (E, E_\bullet) is α -semistable then E is semistable as a vector bundle*

If moreover the rank and degree of E are coprime then E is semistable if and only if it is stable, so the stability of the parabolic vector bundle (E, E_\bullet) is equivalent to the stability of the underlying vector bundle E .

The constant system of weights $\alpha_0 \equiv 0$ lies in the frontier of the concentrated chamber. A parabolic vector bundle is α_0 -stable if its underlying vector bundle is stable. If the rank and degree of E are coprime then the arithmetic wall passing through $\alpha \equiv 0$ cannot be realized in a geometric wall and, therefore, the stability is equivalent of the stability of the underlying vector bundle.

Theorem 4.8.2. *Let X be an irreducible smooth complex projective curve of genus $g \geq 6$ and let D be an irreducible effective divisor over X . Let $r \geq 2$ and let α be a generic concentrated full flag system of weights over D of rank r . Let ξ be a line bundle over X such that $\deg(\xi)$ is coprime with r . Let $\mathcal{M}(r, \alpha, \xi)$ be the moduli space of stable parabolic vector bundles of rank r over (X, D) with system of weights α and determinant ξ . Let $\Phi : \mathcal{M}(r, \alpha, \xi) \rightarrow \mathcal{M}(r, \alpha, \xi)$ be an automorphism. Then there exists a basic transformation T of the form $T = (\sigma, s, L, 0)$ with $T(\xi) \cong \xi$ such that $\Phi = T$. In fact*

$$\text{Aut}(\mathcal{M}(r, \alpha, \xi)) \cong \{T = (\sigma, s, L, 0) \in \mathcal{T} \mid T(\xi) = \xi\} < \mathcal{T}$$

Proof. By Theorem 4.6.24, for every automorphism Φ there exists a basic transformation $T \in \mathcal{T}$ such that $\Phi(E, E_\bullet) = T(E, E_\bullet)$ for all $(E, E_\bullet) \in \mathcal{M}(r, \alpha, \xi)$ and such that

- $T(\xi) \cong \xi$
- $T(\alpha)$ is in the same chamber as α

Let $T = (\sigma, s, L, 0) \in \mathcal{T}$. The pullback of a concentrated system of weights is concentrated and the dual of a concentrated system of weights is concentrated, so $T(\alpha)$ lies in the concentrated chamber for every concentrated α . In particular, this proves that T induces an automorphism whenever $T(\xi) \cong \xi$.

Therefore, it is enough to prove that if $T = (\sigma, s, L, H) \in \mathcal{T}_\xi$ induces an automorphism of the moduli space then $H = 0$. Let $T_0 = (\sigma, s, L, 0)$. Then $T = T_0 \circ \mathcal{H}_H$. We have

$$T_0^{-1} = (\sigma^{-1}, s, \sigma^* L^{-s}, 0)$$

By the previous discussion we know that $T_0^{-1}(\alpha)$ is concentrated, so it induces an isomorphism

$$T_0^{-1} : \mathcal{M}(r, \alpha, \xi) \xrightarrow{\sim} \mathcal{M}(r, \alpha, T_0^{-1}(\xi))$$

composing with Φ we obtain an isomorphism

$$T_0^{-1} \circ \Phi = \mathcal{H}_H : \mathcal{M}(r, \alpha, \xi) \xrightarrow{\sim} \mathcal{M}(r, \alpha, T_0^{-1}(\xi))$$

So for every $(E, E_\bullet) \in \mathcal{M}(r, \alpha, \xi)$, $\mathcal{H}_H(E, E_\bullet)$ must be α -stable. Let $d = \deg(\xi)$. Tensoring with a suitable line bundle we might assume that $0 < d < r$. By hypothesis $\deg(T(\xi)) \cong \xi$. Computing degrees in the determinant equality yields the following possibilities for $|H|$

1. If $s = 1$, then $|H|$ is a positive multiple of r and, therefore, $|H| \geq r > d$
2. If $s = -1$, then $-(d - |H| + kr) = d$, so $|H| = 2d + kr$.
 - (a) If $k \geq 0$ then $|H| \geq 2d > d$.
 - (b) If $k < 0$, then as $d < r$ yields $|H| < 2r + kr = (2 + k)r$. As we assumed $|H| > 0$, then we can only have $k = -1$ and, therefore, $|H| = 2d - r > 0$.

Nevertheless, applying Lemma 4.4.1 in cases (1) and (2a) or Lemma 4.4.3 in case (2b), we deduce that there exists some $(E, E_\bullet) \in \mathcal{M}(r, \alpha, \xi)$ such that $\mathcal{H}_H(E, E_\bullet)$ is α -unstable if $H \neq 0$. \square

Observe that for every $\sigma : X \rightarrow X$ preserving the set D , $\deg(\sigma^*\xi) = \deg(\xi)$. Therefore, there exists a line bundle L_σ such that

$$L^r \otimes \xi \cong (\sigma^{-1})^*\xi$$

on the other hand, $\deg(\sigma^*\xi^{-1}) = -\deg(\xi)$. Therefore, there only exists a line bundle L such that

$$(\sigma, -1, L, 0)(\xi) \cong \xi$$

if $r|2d$. Under the hypothesis that r and d are coprime this can only be attained if $r = 2$. Therefore, for $r > 2$ the automorphisms of $\mathcal{M}(r, \alpha, \xi)$ are the ones generated by pullbacks and tensoring with a line bundle. For every $\sigma : X \rightarrow X$ the set of possible line bundles L such that $\sigma^*(L^r \otimes \xi) \cong \xi$ is in bijection with the r -torsion points of the Jacobian.

Let $L \in J(X)[r]$ be an r -torsion point of the Jacobian. Then for every $\sigma : X \rightarrow X$

$$(\sigma^*L)^r = \sigma^*L^r = \sigma^*\mathcal{O}_X = \mathcal{O}_X$$

so $\sigma^*L \in J(X)[r]$. Therefore $\text{Aut}(X, D)$ is normal in $\text{Aut}(\mathcal{M}(r, \alpha, \xi))$ and we obtain that

$$\text{Aut}(\mathcal{M}(r, \alpha, \xi)) \cong J(X)[r] \rtimes \text{Aut}(X, D)$$

Analogously, for $r = 2$ we obtain

$$\text{Aut}(\mathcal{M}(2, \alpha, \xi)) \cong (J(X)[2] \rtimes \mathbb{Z}_2) \rtimes \text{Aut}(X, D)$$

This is far less than the order of \mathcal{T}_ξ , as for every $\sigma \in \text{Aut}(X, D)$ and for every $0 \leq H < (r-1)D$ and $s \in \{1, -1\}$ such that

$$s(d - |H|) \equiv d \pmod{r}$$

there exists a line bundle L such that

$$(\sigma, s, L, H)(\xi) \cong \xi$$

where $d = \deg(\xi)$. If L' is another line bundle such that $(\sigma, s, L', H)(\xi) \cong \xi$ then there exists an r -torsion point of the Jacobian $S \in J(X)[r]$ such that $L' = L \otimes S$. For any choice of σ and s , the possible divisors H with $0 \leq H < (r-1)D$ are isomorphic to the group $\mathbb{Z}_r^{|D|}$. Nevertheless, if we impose the additional constraint

$$|H| \equiv (1-s)d \pmod{r}$$

Then solutions for $s = 1$ form the subgroup $\mathbb{Z}_r^{|D|-1}$, while any two solutions for $s = -1$ differ by a solution for $s = 0$. Then a direct computation using the relations described in Section 4.4 yields

$$\text{Aut}_{3\text{-Bir}}(\mathcal{M}(r, \alpha, \xi)) \cong \mathcal{T}_\xi \cong \left((J(X)[r] \rtimes \mathbb{Z}_r^{|D|-1}) \rtimes \mathbb{Z}_2 \right) \rtimes \text{Aut}(X, D)$$

Under the coprimality condition, if $|D| > 1$, this group is twice as big as $\text{Aut}(\mathcal{M}(r, \alpha, \xi))$ for $r = 2$ and $2r^{|D|-1}$ times bigger for $r > 2$. This is an example that shows how the combination of the topological constraint $T(\xi) \cong \xi$ and the stability constraint

stating that $T(\alpha)$ and α share the same stability chamber can be really restrictive and reduce the automorphism group $\mathcal{M}(r, \alpha, \xi)$ significantly.

In the concentrated chamber, the stability condition eliminates the Hecke transform \mathcal{H}_H and all its combinations from the possible automorphisms. Topologically, Hecke transformation is the most flexible transformation, in the sense that it is the only one lacking numerical restrictions on the degree of the resulting line bundle. If ξ and ξ' are any two line bundles there exist a line bundle L and a divisor H such that $\mathcal{T}_L \circ \mathcal{H}_H(\xi) = \xi'$. On the other hand, dualization can only pass from degree d line bundles to degree $-d$ and \mathcal{T}_L can only reach line bundles whose degree differs from the original one by a multiple of r .

Therefore, once Hecke transformations are discarded the topological constraint $T(\xi) \cong \xi$ (or, more precisely, the induced numerical constraint $\deg(T(\xi)) = \deg(\xi)$) becomes a really strong condition. This explains the huge difference with respect to \mathcal{T}_ξ . If we allow 2-rational maps, then Hecke transformations are no longer discarded and, therefore, they are available to be used in combination with dualization and tensorization. This flexibilizes the topological constraint $T(\xi) \cong \xi$, leading to more possibilities for the basic transformations $T \in \mathcal{T}_\xi$.

4.9 Stability chamber analysis

From Theorem 4.6.22 we know that every isomorphism between two moduli spaces of parabolic vector bundles is induced by some basic transformation. In particular, in Theorem 4.6.24 we proved that the automorphism group of $\mathcal{M}(r, \alpha, \xi)$ is the subgroup of \mathcal{T} consisting on basic transformations such that

- $T(\xi) \cong \xi$
- $T(\alpha)$ belongs to the same stability chamber as α

As we mentioned in the last section, the first condition is computable and purely topological, but the second one is of a different kind. Determining whether two parabolic weights α and α' over the same curve (X, D) belong to the same stability chamber is highly nontrivial and depends greatly on the geometry of the curve X . Two systems of weights α and α' belong to different stability chambers if and only if there exists some α -stable parabolic vector bundle (E, E_\bullet) which is α' -unstable or viceversa, i.e., if there exists some α' -stable parabolic vector bundle which is α -unstable.

Assume that (E, E_\bullet) is α -stable but α' -unstable. Then there exists a maximal destabilizing subsheaf $F \subset E$ such that

$$\frac{\text{pardeg}_{\alpha'}(F, F_\bullet)}{\text{rk}(F)} > \frac{\text{pardeg}_{\alpha'}(E, E_\bullet)}{\text{rk}(E)}$$

but, from α -stability

$$\frac{\text{pardeg}_{\alpha}(F, F_\bullet)}{\text{rk}(F)} < \frac{\text{pardeg}_{\alpha}(E, E_\bullet)}{\text{rk}(E)}$$

therefore, the existence of a destabilizing subsheaf imposes some numerical conditions on α , α' and the topological invariants of (E, E_\bullet) and (F, F_\bullet) . If this numerical

conditions are not satisfied by α and α' then it is clear that they belong to the same stability chamber. In this case we say that α and α' belong to the same numerical chamber.

Nevertheless, the reciprocal is not always true. Even if α and α' satisfy the numerical conditions which are necessary for the existence of a destabilizing subbundle, finding a parabolic vector bundle (E, E_\bullet) and a subsheaf $F \subset E$ with the needed invariants is not obvious. In fact, there might exist systems of weights α and α' such that the numerical conditions allowed the existence of α -stable and α' -unstable parabolic vector bundles but such that geometrically there do not exist at all. Therefore, the stability chambers are divided in several numerical chambers whose walls are not realized geometrically by any parabolic vector bundle.

We will start identifying some numerical invariants that will allow us to determine the numerical chambers uniquely.

Let $\{n_1(x), \dots, n_r(x)\} = \bar{n}$ be any set of nonnegative integers. We say that \bar{n} is admissible if for any $i = 1, \dots, r$ and any $x \in D$, $n_i(x) \in \{0, 1\}$ and there exists $0 < r' < r$ such that for all $x \in D$ yields $\sum_{i=1}^r n_i(x) = r'$. Let $d = \deg(\xi)$. We define

$$M(r, \alpha, d, \bar{n}) = \left\lfloor \frac{r'd + r' \sum_{x \in D} \sum_{i=1}^r \alpha_i(x) - r \sum_{x \in D} \sum_{i=1}^r n_i(x) \alpha_i(x)}{r} \right\rfloor \in \mathbb{Z}$$

Observe that for every $\varepsilon \in \mathbb{R}^{|D|}$,

$$M(r, \alpha, d, \bar{n}) = M(r, \alpha[\varepsilon], d, \bar{n})$$

i.e., $M(r, \alpha, d, \bar{n})$ only depends on the class $\alpha \in \tilde{\Delta}$.

Recall that we say that if a subbundle $F \subsetneq E$ of a parabolic vector bundle (E, E_\bullet) is of type \bar{n} then

$$\text{wt}(F, F_\bullet) = \sum_{x \in D} \sum_{i=1}^r n_i(x) \alpha_i(x)$$

Lemma 4.9.1. *Let (E, E_\bullet) be a parabolic vector bundle such that $\deg(E) = d$. Then (E, E_\bullet) is semistable if and only if for every admissible \bar{n} and every subbundle $F \subsetneq E$ of type \bar{n} we have*

$$\deg(F) \leq M(r, \alpha, d, \bar{n})$$

Proof. The parabolic bundle (E, E_\bullet) is semistable if for every subbundle F with the induced parabolic structure

$$\frac{\deg(F) + \sum_{x \in D} \sum_{i=1}^r n_i(x) \alpha_i(x)}{r'} \leq \frac{d + \sum_{x \in D} \sum_{i=1}^r \alpha_i(x)}{r}$$

Equivalently, solving for $\deg(F)$

$$\deg(F) \leq \frac{r'd + r' \sum_{x \in D} \sum_{i=1}^r \alpha_i(x) - r \sum_{x \in D} \sum_{i=1}^r n_i(x) \alpha_i(x)}{r}$$

As $\deg(F)$ is an integer, its value is at most the floor of the right hand side, which is precisely $M(r, \alpha, d, \bar{n})$. \square

Corollary 4.9.2. *Let α and α' be rank r systems of weights such that for every admissible \bar{n}*

$$M(r, \alpha, d) = M(r, \alpha', d)$$

then α and α' belong to the same stability chamber.

Proof. If (E, E_\bullet) is α -semistable then for every admissible \bar{n} and every subbundle $F \subset E$

$$\deg(F) \leq M(r, \alpha, d) = M(r, \alpha', d)$$

so (E, E_\bullet) is α' -semistable. \square

Let \mathcal{N} be the set of admissible \bar{n} . Let us denote

$$\overline{M}(r, \alpha, d) = (M(r, \alpha, d, \bar{n}))_{\bar{n} \in \mathcal{N}} \in \mathbb{Z}^{\mathcal{N}}$$

then we say that α and α' belong to the same numerical stability chamber if and only if $\overline{M}(r, \alpha, d) = \overline{M}(r, \alpha', d)$.

Proposition 4.9.3. *There is a finite number of stability chambers in Δ .*

Proof. For every $\alpha \in \Delta$ and every admissible \bar{n} , using that $0 \leq \alpha_i(x) < 1$ and $0 \leq n_i(x) < 1$ we obtain the following bounds

$$\begin{aligned} M_{\min}(r, d) = \frac{d}{r} - r|D| - 1 &< \left\lfloor \frac{r'd + r' \sum_{x \in D} \sum_{i=1}^r \alpha_i(x) - r \sum_{x \in D} \sum_{i=1}^r n_i(x) \alpha_i(x)}{r} \right\rfloor \\ &\leq \frac{(r-1)d}{r} + (r-1)|D| = M_{\max}(r, d) \end{aligned}$$

Therefore $\overline{M}(r, \alpha, d) \in [M_{\min}(r, d), M_{\max}(r, d)]^{\mathcal{N}}$ for every α . In particular this implies that there is a finite number of numerical chambers in Δ . As a numerical chamber is included in exactly one stability chamber we obtain that there is a finite number of stability chambers. \square

This proposition has some further implications on the k -birational geometry of the moduli space $\mathcal{M}(r, \alpha, \xi)$.

Corollary 4.9.4. *Let $k > 0$. Let X be a genus $g \geq 1 + \frac{l-1}{r-1}$ Riemann surface and let $D \subset X$ be a nonempty set of points. Let α be any generic system of weights over (X, D) and let ξ be any line bundle over X . Then there exists an open subset $\mathcal{M}^{\text{us}}(r, \xi) \subset \mathcal{M}(r, \alpha, \xi)$ whose complement has codimension at least k and such that each parabolic vector bundle $(E, E_\bullet) \in \mathcal{M}^{\text{us}}(r, \xi)$ is α' -stable for every generic $\alpha' \in \Delta$.*

Proof. Let \mathcal{C} denote the set of stability chambers in Δ . By the previous lemma it is a finite set. Let $\alpha_1, \dots, \alpha_{|\mathcal{C}|}$ be a set of generic representatives for the stability chambers in \mathcal{C} . Then a parabolic vector bundle is α' -stable for all generic $\alpha' \in \Delta$ if and only if it is α_i -stable for every $i = 1, \dots, |\mathcal{C}|$. On the other hand by Corollary 4.1.4, for every α_i there exists an open subset $\mathcal{U}_i \subset \mathcal{M}(r, \alpha, \xi)$ whose

complement has codimension at least k such that every $(E, E_\bullet) \in \mathcal{U}_i$ is α -stable and α_i -stable. Take

$$\mathcal{M}^{\text{us}}(r, \xi) = \bigcap_{i=1}^{|\mathcal{C}|} \mathcal{U}_i \subset \mathcal{M}(r, \alpha, \xi)$$

As \mathcal{C} is a finite set, $\mathcal{M}^{\text{us}}(r, \xi)$ is an open subset whose complement has codimension at least 2 and such that every $(E, E_\bullet) \in \mathcal{M}^{\text{us}}(r, \xi)$ is α_i -stable for every $i = 1, \dots, |\mathcal{C}|$. \square

Once we have classified the space of numerical chambers, our objective is to develop a tool to determine whether some numerical barrier separating two numerical chambers is actually realized by a destabilizing subbundle of some parabolic vector bundle, at least for big genus.

Lemma 4.9.5. *Let X be a genus g smooth complex projective curve. Suppose that*

$$g \geq 1 + (r-1)n - \left\lfloor \sum_{x \in D} \alpha_1(x) \right\rfloor$$

Then for every \bar{n} there exist a stable parabolic vector bundle $(E, E_\bullet) \in \mathcal{M}(r, \alpha, \xi)$ and a subbundle $F \subsetneq E$ of type \bar{n} such that

$$\deg(F) = M(r, \alpha, d, \bar{n})$$

Proof. For every admissible choice of \bar{n}

$$\sum_{x \in D} \sum_{i=1}^r n_i(x) \alpha_i(x) \geq \sum_{x \in D} \alpha_1(x)$$

Therefore, the genus condition in [BB05, Theorem 1.4.3A] hold for every \bar{n} and we obtain that there exists a stable parabolic vector bundle (E, E_\bullet) of rank r and degree $d = \deg(\xi)$ with a subbundle $F \subsetneq E$ satisfying the properties in the Lemma. Now it is enough to tensor it with a suitable degree zero line bundle to obtain another one whose determinant is isomorphic to ξ . \square

Theorem 4.9.6. *Let α and β be generic full flag systems of weights of rank r over (X, D) . Let ξ be a degree d line bundle over X and assume that*

$$g \geq 1 + (r-1)n - \min \left(\left\lfloor \sum_{x \in D} \alpha_1(x) \right\rfloor, \left\lfloor \sum_{x \in D} \beta_1(x) \right\rfloor \right)$$

Then α and β belong to the same stability chamber of the moduli space of rank r determinant ξ full flag parabolic vector bundles if and only if for every admissible \bar{n}

$$M(r, \alpha, d, \bar{n}) = M(r, \beta, d, \bar{n})$$

Proof. The systems of weights α and β belong to different chambers if and only if either there exists an α -stable vector bundle (E, E_\bullet) which is not β -stable or vice

versa. Suppose that there exists an α -stable, β -unstable parabolic vector bundle. By Lemma 4.9.1, there exist a subbundle $F \subsetneq E$ and integers \bar{n} such that

$$\text{wt}(F, F_\bullet) = \sum_{x \in D} \sum_{i=1}^r n_i(x) \alpha_i(x)$$

and

$$M(r, \beta, d, \bar{n}) < \deg(F) \leq M(r, \alpha, d, \bar{n})$$

so $M(r, \beta, d, \bar{n}) \neq M(r, \alpha, d, \bar{n})$. Reciprocally, suppose that $\overline{M}(r, \alpha, d) \neq \overline{M}(r, \beta, d)$. Then, interchanging α and β if necessary, there exists an admissible \bar{n} such that $M(r, \beta, d, \bar{n}) < M(r, \alpha, d, \bar{n})$. By Lemma 4.9.5, there exist an α -stable parabolic vector bundle (E, E_\bullet) and a subbundle $F \subsetneq E$ of type \bar{n} such that

$$\deg(F) = M(r, \alpha, d, \bar{n}) > M(r, \beta, d, \bar{n})$$

Therefore, from Lemma 4.9.1, (E, E_\bullet) is β -unstable. \square

The genus condition in this Theorem deserves some remarks. First, notice that it is only needed for the “necessary” part of the theorem. If $\overline{M}(r, \alpha, d) = \overline{M}(r, \beta, d)$ then α and β belong to the same numerical – and therefore geometrical – chamber, independently of the genus of the curve.

Second, the genus condition is picked so that it is valid for any couple of systems of weights α and β . There are stability chambers which are more easily distinguished than others. For some choices of α and β , the bound for the genus can be really lowered.

Proposition 4.9.7. *Let α and β be concentrated systems of weights and let \bar{n} be an admissible array such that*

$$M(r, \beta, d, \bar{n}) < M(r, \alpha, d, \bar{n})$$

Then α and β belong to different stability chambers if

$$g \geq 1 + \frac{\lfloor \sum_{x \in D} \sum_{i=1}^r (1 - \alpha_i(x))(1 - n_i(x)) \rfloor}{r'}$$

Proof. The proof is exactly the same as in the Theorem, but instead of using the genus bound in Lemma 4.9.5, we apply the bound in [BB05, Theorem 1.4.3A]. \square

Finally, observe that the genus bounds for the previous results are not well defined for $\alpha, \beta \in \tilde{\Delta}$, rather they depend on the choice of representatives in Δ . We can play this out in our favor and choose suitable $\varepsilon, \delta \in \mathbb{R}^{|D|}$ such that the genus bound for $\alpha[\varepsilon]$ and $\beta[\delta]$ is as low as possible. The bound for $\alpha[\varepsilon]$ decreases with ε . The maximum possible shift that we can take at each $x \in D$ is $\varepsilon(x) < 1 - \alpha_r(x)$. Therefore, the previous Lemma hold if for some $\tau > 0$

$$g \geq 1 + \frac{\lfloor \sum_{x \in D} \sum_{i=1}^r (\alpha_r(x) - \tau - \alpha_i(x))(1 - n_i(x)) \rfloor}{r'}$$

In particular, the more concentrated the weights in a numerical chamber are, the lesser genus is needed in order to realize the surrounding numerical barriers as

geometrical barriers. This somehow justifies that our study of the concentrated chamber can be done more explicitly in lower genus.

Finally, we can apply the previous results to obtain the following versions of Theorem 4.6.22 and Theorem 4.6.24.

Theorem 4.9.8. *Let (X, D) and (X', D') be two smooth projective curves of genus $g \geq \max\{1 + (r - 1)|D|, 6\}$ and $g' \geq 6$ respectively with set of marked points $D \subset X$ and $D' \subset X'$. Let ξ and ξ' be line bundles over X and X' respectively, and let α and α' be full flag generic systems of weights over (X, D) and (X', D') respectively. Let*

$$\Phi : \mathcal{M}(X, r, \alpha, \xi) \xrightarrow{\sim} \mathcal{M}(X', r', \alpha', \xi')$$

be an isomorphism. Then

1. $r = r'$
2. (X, D) is isomorphic to (X', D') , i.e., there exists an isomorphism $\sigma : X \xrightarrow{\sim} X'$ sending D to D' .
3. There exists a basic transformation T such that
 - $\sigma^*\xi' \cong T(\xi)$
 - $\overline{M}(r, \sigma^*\alpha', \deg(\xi')) = \overline{M}(r, T(\alpha), \deg(\xi'))$
 - For every $(E, E_\bullet) \in \mathcal{M}(r, \alpha, \xi)$, $\sigma^*\Phi(E, E_\bullet) \cong T(E, E_\bullet)$

Corollary 4.9.9. *Let (X, D) be a smooth projective curve of genus $g \geq \max\{1 + (r - 1)|D|, 6\}$ and let α be a full flag generic system of weights over (X, D) of rank r . Let ξ be a line bundle over X . Then the automorphism group of $\mathcal{M}(r, \alpha, \xi)$ is the subgroup of \mathcal{T} consisting on basic transformations T such that*

- $T(\xi) \cong \xi$
- $\overline{M}(r, T(\alpha), \deg(\xi)) = \overline{M}(r, \alpha, \deg(\xi))$

Unlike the original results, these versions are fully computable for each specific case, in the sense that for every system of weights α and every line bundle ξ we have an explicit morphism

$$\begin{aligned} (\det_\xi, \overline{M}_\alpha) : \mathcal{T} &\longrightarrow \text{Pic}(X) \times \mathbb{Z}^N \\ T &\longmapsto (T(\xi), \overline{M}(r, T(\alpha), \deg(T(\xi)))) \end{aligned}$$

And for $g \geq 1 + (r - 1)|D|$ we know that the set of isomorphisms between $\mathcal{M}(r, \alpha, \xi)$ and $\mathcal{M}(r, \alpha', \xi')$ is given by

$$(\det_\xi, \overline{M}_\alpha)^{-1}(\xi', \overline{M}(r, \alpha', \deg(\xi')))$$

In particular, the moduli spaces $\mathcal{M}(r, \alpha, \xi)$ and $\mathcal{M}(r, \alpha', \xi)$ are isomorphic if and only if

$$(\xi', \overline{M}(r, \alpha', \deg(\xi'))) \in (\det_\xi, \overline{M}_\alpha)(\mathcal{T})$$

Moreover, from the description of \mathcal{T} in terms of the generators \mathcal{D}^- , \mathcal{T}_L and \mathcal{H}_H given in Proposition 4.4.9

$$\mathcal{T} \cong \langle \mathcal{T}_L, \mathcal{T}_H \rangle \rtimes (\text{Aut}(X, x) \times \mathbb{Z}/2\mathbb{Z})$$

for each chamber α and each determinant ξ we can explicitly describe a presentation of

$$\text{Aut}(\mathcal{M}(r, \alpha, \xi)) = (\det_\xi, \overline{M}_\alpha)^{-1}(\xi, \overline{M}(r, \alpha, \deg(\xi))) < \mathcal{T}$$

just by selecting generators in the right hand side.

4.10 Examples

Let X be a curve with an automorphism $\sigma : X \rightarrow X$ such that there exist $x, y \in X$ with $\sigma(x) = y$ and $\sigma(y) = x$. Take $D = \{x, y\}$. Let $0 \leq \alpha_1 < 1/2 < \alpha_2 < 1$. Then take the following full flag system of weights of rank $r = 2$ at (X, D)

$$\begin{aligned} \alpha_1(x) &= \alpha_1 \\ \alpha_2(x) &= \alpha_2 \\ \alpha_1(y) &= \alpha_2 - 1/2 \\ \alpha_2(y) &= \alpha_1 + 1/2 \end{aligned}$$

Then, by construction $\mathcal{H}_{x+y}(\alpha) \sim \Sigma_\sigma(\alpha)$. Let L be a line bundle of degree 1 such that $L^2 \cong \mathcal{O}_X(x + y)$ Then we have that

$$(\sigma, 1, L, x + y) : \mathcal{M}(r, \alpha, \xi) \longrightarrow \mathcal{M}(r, \alpha, \xi)$$

is an automorphism. If we additionally took the weights α_i so that $\alpha_1 + \alpha_2 = 1$, then $\mathcal{D}^-(\alpha) = \alpha$. Nevertheless, we can prove that there always exists a system α' in the same chamber as α with $\mathcal{D}^-(\alpha') = \alpha'$. Let

$$\begin{aligned} \alpha_1 &= 1/2 - \epsilon_1 \\ \alpha_2 &= 1/2 + \epsilon_2 \end{aligned}$$

Take $\delta = \frac{\epsilon_1 - \epsilon_2}{2}$. Then, taking

$$\begin{aligned} \alpha'_1(x) = \alpha_1(x) + \delta &= \frac{1}{2} - \frac{\epsilon_1 + \epsilon_2}{2} \\ \alpha'_2(x) = \alpha_2(x) + \delta &= \frac{1}{2} + \frac{\epsilon_1 + \epsilon_2}{2} \\ \alpha'_1(y) = \alpha_1(y) + \delta &= \frac{\epsilon_1 + \epsilon_2}{2} \\ \alpha'_2(y) = \alpha_2(y) + \delta &= 1 - \frac{\epsilon_1 + \epsilon_2}{2} \end{aligned}$$

As $0 < \epsilon_i < \frac{1}{2}$ for $i = 1, 2$, then $-\frac{1}{2} < \delta < \frac{1}{2}$. As all the resulting weights are between 0 and 1, then α' is a system of weights in the same stability chamber as α .

Now let

$$\begin{aligned} \text{Aut}^+(X, D) &= \{\sigma \in \text{Aut}(X) | \sigma(x) = x, \sigma(y) = y\} \\ \text{Aut}^-(X, D) &= \{\sigma \in \text{Aut}(X) | \sigma(x) = y, \sigma(y) = x\} \end{aligned}$$

Then the following basic transformations are automorphisms of $\mathcal{M}(r, \alpha, \xi)$

- $T = (\sigma^+, s, L, x + y)$, where $\sigma^- \in \text{Aut}^-(X, D)$ and $T(\xi) \cong \xi$
- $T = (\sigma^-, s, L, 0)$, where $\sigma^+ \in \text{Aut}^+(X, D)$ and $T(\xi) \cong \xi$.

Moreover, if $|\delta|$ is small enough and X has genus $g \geq 3$, then the weights $\alpha_i(x)$ are concentrated but the weights $\alpha_i(y)$ are not. Therefore, $\mathcal{H}_y(\alpha)$ is concentrated, α is concentrated at x , \mathcal{H}_{x+y} is concentrated at y and $\mathcal{H}_x(\alpha)$ is not concentrated. From the genus condition, it can be proved using Theorem 4.9.6 from the last section, that $\mathcal{H}_{x+y}(\alpha)$, $\mathcal{H}_x(\alpha)$ and $\mathcal{H}_y(\alpha)$ do not belong to the same chamber as α . Moreover, taking the pullback by σ^- interchange the following (distinct) chambers

- $\mathcal{H}_x(\alpha)$ and $\mathcal{H}_y(\alpha)$
- α and $\mathcal{H}_{x+y}(\alpha)$

As all the chambers are different, in order for a basic transformation $T = (\sigma, s, L, H)$ to preserve the stability chamber of α we need either

- $\sigma \in \text{Aut}^+(X, D)$ and $H = 0$ or
- $\sigma \in \text{Aut}^-(X, D)$ and $H = x + y$

so we obtain that the automorphisms of $\mathcal{M}(r, \alpha, \xi)$ are precisely the ones described above.

This example proves that there exist curves and systems of weights for which the Hecke transform induces nontrivial automorphisms when combined with pullbacks by suitable automorphisms of the curve even if the transformation \mathcal{H}_H alone does not preserve the stability chamber.

As we saw in the last theorem, this cannot happen in the concentrated setting and, in general, it is not expected to happen if the parabolic chamber is stable under transformations Σ_σ for all $\sigma \in \text{Aut}(X, D)$.

Now let X be any Riemann surface and let $D = x$ for some $x \in X$. Let $0 < \varepsilon < 1/4$ and let us consider the following rank 3 system of weights over (X, D)

$$\begin{aligned}\alpha_1(x) &= \varepsilon \\ \alpha_2(x) &= 3\varepsilon \\ \alpha_3(x) &= 1 - \varepsilon\end{aligned}$$

A direct computation shows us that $\mathcal{H}_x(\alpha) \sim (\varepsilon, 1 - 3\varepsilon, 1 - \varepsilon)$, so $\mathcal{H}_x(\alpha)^\vee \sim \alpha$. Let ξ be any degree -1 line bundle over X . Then

$$\mathcal{D}^- \circ \mathcal{H}_x(\xi) = (\xi(-x))^{-1} = \xi^{-1}(x)$$

so $\deg(\mathcal{D}^- \circ \mathcal{H}_x(\xi)) = 1 + 1 = 2 = \deg(\xi) + 3$. Therefore, there exists a line bundle L of degree 1 such that

$$L^3 \otimes \xi(-x) \cong \xi^{-1}$$

Take $T = (\text{Id}, -1, L, x)$. As \mathcal{T}_L does not change the parabolic weights the previous computations shows that

- $T(\xi) = \xi$

- $T(\alpha) \sim \alpha$

Therefore, we obtain that

$$(\text{Id}, -1, L, x) : \mathcal{M}(r, \alpha, \xi) \longrightarrow \mathcal{M}(r, \alpha, \xi)$$

is an automorphism. Moreover, for any automorphism $\sigma : X \longrightarrow X$ fixing $D = x$ we have that

$$\deg(\mathcal{D}^- \circ \mathcal{H}_x(\xi)) = 2 = \deg((\sigma^{-1})^* \xi) + 3$$

Therefore, there exists a line bundle L_σ of degree 1 such that

$$L_\sigma^3 \otimes \xi(-x) \cong (\sigma^{-1})^* \xi^{-1}$$

As Σ_σ fixes the parabolic point then taking $T = (\sigma, -1, L_\sigma, x)$ we obtain that

- $T(\xi) = \xi$
- $T(\alpha) \sim \alpha$

Therefore, we obtain that

$$(\sigma, -1, L_\sigma, x) : \mathcal{M}(r, \alpha, \xi) \longrightarrow \mathcal{M}(r, \alpha, \xi)$$

is an automorphism. Then we have found an example of a marked curve of arbitrary high genus and a system of weights such that the Hecke transformation induces a nontrivial automorphism of the moduli space when combined with the dualization. In contrast with the previous example, where the curve was supposed to have an automorphism interchanging two parabolic points, in this example the existence of an automorphism involving Hecke transformation is achieved even if the curve is generic and lacks nontrivial automorphisms.

The basic transformation $T = (\text{Id}, -1, L, x)$ is particularly interesting. If $g \geq 4$ then from Lemma 4.6.23 we know that T acts nontrivially on $\mathcal{M}(r, \alpha, \xi)$, but a direct computation shows that $T^2 = \text{Id}_{\mathcal{T}}$. Therefore, T is an involution of $\mathcal{M}(r, \alpha, \xi)$ that does not come from an involution of the Riemann surface X .

To complete the example, let us study other kinds of automorphisms that this moduli space admits. Let $T = (\sigma, s, L, H) \in \mathcal{T}$. By construction $\mathcal{D}^-(\alpha) \sim \mathcal{H}_x(\alpha)$. Moreover, if ϵ is small enough then $\mathcal{H}_{2x}(\alpha) \sim (1 - 5\epsilon, 1 - 3\epsilon, 1 - \epsilon)$ is concentrated. Therefore, so is $\mathcal{D}^- \circ \mathcal{H}_{2x}(\alpha)$. On the other hand, α and $\mathcal{H}_x(\alpha)$ are not concentrated. Using the results of the previous chapter we can prove that if ϵ is small enough and $g \geq 3$ then $\alpha \sim \mathcal{D}^- \circ \mathcal{H}_x(\alpha)$, $\mathcal{H}_x(\alpha) \sim \mathcal{D}^-(\alpha)$ and $\mathcal{H}_{2x}(\alpha) \sim \mathcal{D}^- \circ \mathcal{H}_{2x}(\alpha)$ belong to three different stability chambers.

On the other hand, Σ_σ and \mathcal{T}_L do not change the stability chamber, so $T(\alpha)$ is in the same stability chamber as α if and only if either

- $H = 0$ and $s = 1$ or
- $H = x$ and $s = -1$

In both cases, for every $\sigma : X \rightarrow X$ fixing $D = x$ there exists a line bundle L such that $(\sigma, 1, L, 0)(\xi) \cong \xi$ or $(\sigma, -1, L, x)(\xi) \cong \xi$ respectively. In every case, such L is unique up to a choice of a 3-torsion point in $J(X)$. Then

$$\mathrm{Aut}(\mathcal{M}(r, \alpha, \xi)) \cong J(X)[3] \rtimes \mathbb{Z}^2 \rtimes \mathrm{Aut}(X, D)$$

An analogous example can be found for any rank. Just take α distributed as $\alpha_r(x) = 1 - \varepsilon$ and $\alpha_k(x) = (2k - 1)\varepsilon$ for $k < r$. Then $\mathcal{D}^- \circ \mathcal{H}_{(r-2)x}(\alpha) \sim \alpha$. If we take ξ of degree -1 then

$$\deg(\mathcal{H}_{(r-2)x}(\xi)) = \deg(\xi) - r + 2 = r - 1 = \deg(\xi^{-1}) + r$$

Therefore, there exists a line bundle L of degree 1 such that if $T = (\mathrm{Id}, -1, L, \mathcal{H}_{(r-2)x})$ then

- $T(\alpha) \sim \alpha$
- $T(\xi) = \xi$

so T induces an automorphism $T : \mathcal{M}(r, \alpha, \xi) \rightarrow \mathcal{M}(r, \alpha, \xi)$ which is an involution of the moduli space.

4.11 Further comments, results and applications

In this section I would like to point out some conclusions about the geometry of the moduli spaces of parabolic vector bundles obtained through the study of the isomorphisms and k -birational equivalences between the moduli spaces, as well as stating some applications of the previous work.

4.11.1 Refined Torelli type theorems

First of all, I would like to analyze our main classification Theorem 4.6.22 in the context of Torelli type theorems. When we analyze the isomorphisms between moduli spaces of parabolic vector bundles, we observe three related but different problems

1. Find whether $\mathcal{M}(X, r, \alpha, \xi)$ and $\mathcal{M}(X', r', \alpha', \xi')$ are isomorphic for different choices
2. If they are isomorphic, classify the isomorphisms $\Phi : \mathcal{M}(X, r, \alpha, \xi) \rightarrow \mathcal{M}(X', r', \alpha', \xi')$
3. Find the automorphism group of $\mathcal{M}(X, r, \alpha, \xi)$

If we knew the first part, the second one would be completely equivalent to finding the automorphism group of some (and thus both) of the moduli spaces involved. Clearly, if $\Phi : \mathcal{M} \rightarrow \mathcal{M}'$ and $\Phi' : \mathcal{M} \rightarrow \mathcal{M}'$ are two isomorphisms between moduli spaces then $\Phi^{-1} \circ \Phi' : \mathcal{M} \rightarrow \mathcal{M}$ is an automorphism of \mathcal{M} . If we know the automorphisms, then we classify the isomorphisms. Nevertheless, this way of classifying the isomorphisms between the moduli spaces is quite incomplete. We depend on knowing precisely at least one of such isomorphisms, say $\Phi_0 : \mathcal{M} \rightarrow \mathcal{M}'$, to relate the rest of them to the automorphism group.

Conversely, in our theorem we address the classification problem as a whole, treating at the same time the classification of automorphisms and finding the explicit possible isomorphisms between the moduli spaces. This last part, in turn, has some consequences regarding problem (1). From Theorem 4.3.6, we know that if the moduli spaces $\mathcal{M}(X, r, \alpha, \xi)$ and $\mathcal{M}(X', r', \alpha', \xi')$ are isomorphic, then $r = r'$ and $(X, D) \cong (X', D')$. On the other hand, the latter result given in Theorem 4.6.22 provides additional restrictions. If we pick α and α' generically of the same rank r over a pointed curve (X, D) and we chose ξ and ξ' over X , then $\mathcal{M}(X, r, \alpha, \xi)$ and $\mathcal{M}(X, r, \alpha', \xi')$ are probably not isomorphic, as it would be necessary to exist a basic transformation $T \in \mathcal{T}$ sending

$$\begin{aligned} T(\xi) &\cong \xi' \\ T(\alpha) &\sim_{\text{st}} \alpha' \end{aligned}$$

As we commented through Section 4.8, these two conditions are heavily nontrivial and impose different kind of restrictions on the geometry of the moduli spaces. The first one can be reduced to a numerical problem, as we can always shift between line bundles of the same degree without changing the stability conditions simply tensoring with an appropriate degree zero line bundle. If $d = \deg(\xi)$ and $d' = \deg(\xi')$ then if $T = (\sigma, s, L, H)$ we must have

$$d' = s(r \deg(L) + d - |H|)$$

As the choice of L does not affect the stability at all, we can choose $\deg(L) \in \mathbb{Z}$ freely, so this condition really becomes a congruence relation

$$d' \cong s(d - |H|) \pmod{r}$$

On the other hand, the second condition involves an analysis on the parabolic chambers that are reachable by basic transformations similar to the one described in Section 4.9. We can then portrait the parameter space for the moduli $\mathcal{M}(X, r, \alpha, \xi)$ in the following way. Let $\Theta = \mathbb{Z} \times \tilde{\Delta}$, where $\tilde{\Delta}$ is the stability space for the moduli introduced in Section 4.4 and studied through Section 4.9. Then we can define an equivalence relation \sim_{Θ} in Θ as follows. We say that $(d, \alpha) \sim_{\Theta} (d', \alpha')$ if there exists an automorphism $\sigma \in \text{Aut}(X, D)$, a sign $s \in \{-1, 1\}$ and a divisor $0 \leq H \leq (r-1)D$ such that

$$d' \cong s(d - |H|) \pmod{r}$$

$$(\sigma, s, \mathcal{O}_X, H)(\alpha) \text{ belongs to the same stability chamber as } \alpha'$$

More abstractly, let $\tilde{\tilde{\Delta}}$ be the space of stability chambers in $\tilde{\Delta}$, i.e., the quotient of $\tilde{\Delta}$ by the relation $\alpha \sim_{\text{st}} \alpha'$ if and only if α and α' belongs to the same stability chamber. Let $\tilde{\Theta} = \mathbb{Z} \times \tilde{\tilde{\Delta}}$. We know \mathcal{T} acts on Θ in a clear way. Moreover, we know that for every basic transformation $T \in \mathcal{T}$ and every $\alpha \in \tilde{\Delta}$, a parabolic vector bundle (E, E_{\bullet}) is α -stable if and only if $T(E, E_{\bullet})$ is $T(\alpha)$ -stable. Therefore, if α and α' belong to the same stability chamber, then $T(\alpha)$ and $T(\alpha')$ belong to the same stability chamber, so \mathcal{T} acts on the quotient $\tilde{\tilde{\Delta}}$. With this construction it is then straightforward to check that $\Theta / \sim_{\Theta} = \tilde{\Theta} / \mathcal{T}$. Then we can reformulate Theorem 4.6.22 in the following (slightly weaker) way

Theorem 4.11.1. *Let (X, D) and (X', D') be two smooth projective curves of genus $g \geq 6$ and $g' \geq 6$ respectively with set of marked points $D \subset X$ and $D' \subset X'$. Let ξ and ξ' be line bundles over X and X' respectively, and let α and α' be full flag generic systems of weights over (X, D) and (X', D') respectively. Then $\mathcal{M}(X, r, \alpha, \xi) \cong \mathcal{M}(X', r', \alpha', \xi')$ if and only if*

1. $r = r'$
2. (X, D) is isomorphic to (X', D') , i.e., there exists an isomorphism $\sigma : X \xrightarrow{\sim} X'$ sending D to D' .
3. In $\tilde{\Theta}/\mathcal{T}$ we have $(\deg(\xi), \alpha) \sim_{\Theta} (\deg(\xi'), \sigma^* \alpha')$

In other words, we obtain a refined form of a Torelli theorem. The usual Torelli type theorems state that the isomorphism class of a certain moduli space $\mathcal{M}(X)$ identifies unequivocally the isomorphism class of X . In this case, we can interpret this theorem as proving that the isomorphism class of the moduli space $\mathcal{M}(X, r, \alpha, \xi)$ allows us to recover

1. The isomorphism class of the marked curve (X, D)
2. The rank r
3. The orbit $[(\deg(\xi), \alpha)] \in \tilde{\Theta}/\mathcal{T}$

And, in this case, it is an equivalence, i.e., the isomorphisms classes of moduli spaces of parabolic vector bundles for generic parabolic weights are classified by these three data.

In particular, if X has genus $g \geq 1 + (r-1)|D|$, then we can give a further explicit numerical description of the stability chambers, leading to the more computable version of the quotient $\tilde{\Theta}/\mathcal{T}$ and, therefore, of the space of isomorphism classes of the moduli spaces of parabolic vector bundles as we can simply write

$$(d, \alpha) \sim_{\Theta} (d', \alpha') \iff \exists(\sigma, s, H) \left\{ \begin{array}{l} d' \cong s(d - |H|) \pmod{r} \\ \overline{M}(r, \alpha, d) = \overline{M}(r, (\sigma, s, \mathcal{O}_X, H)(\alpha), d') \end{array} \right.$$

Using this point of view, the results in Section 4.8 can be seen as computing the intersection of the orbits $\mathcal{T} \cdot (d, \alpha)$ with the concentrated chamber for α concentrated.

This type of working methodology can be also carried to the non-parabolic situation, leading to similar “strong Torelli” type results. In particular, working with the proof by Biswas, Gómez and Muñoz [BGM13] we can obtain the following result

Theorem 4.11.2. *Let X and X' be two smooth projective curves of genus $g \geq 4$ and $g' \geq 4$ respectively with line bundles ξ and ξ' over X and X' respectively. Then $\mathcal{M}(X, r, \xi) \cong \mathcal{M}(X', r', \xi')$ if and only if*

1. $r = r'$
2. $X \cong X'$
3. $\deg(\xi) \cong \pm \deg(\xi') \pmod{r}$

Proof. If $r = r'$, $\sigma : X \rightarrow X'$ is an isomorphism and $\deg(\xi) \cong s \deg(\xi') \pmod r$ for $s \in \{1, -1\}$, then $\deg(\sigma^* \xi' \otimes \xi^{-1}) \cong 0 \pmod r$, so there exists a line bundle L such that

$$\sigma^* \xi' = (L^r \otimes \xi)^s$$

Therefore, the map

$$E \mapsto \sigma^{-1}(E \otimes L)^s$$

is an isomorphism between $\mathcal{M}(X, r, \xi)$ and $\mathcal{M}(X', r, \xi')$.

On the other hand, if $\Phi : \mathcal{M}(X, r, \xi) \rightarrow \mathcal{M}(X', r', \xi')$ is an isomorphism, then applying [BGM13, Lemma 4.1] there exists a \mathbb{C}^* -equivariant isomorphism $f : \bigoplus_{j=2}^r H^0(X, K_X^j) \rightarrow \bigoplus_{j=2}^{r'} H^0(X', K_{X'}^j)$ on the corresponding Hitchin spaces with the canonical \mathbb{C}^* -action induced by dilations on the cotangent bundle of the moduli space. Therefore, the map f must preserve the filtrations given by the weighted action of \mathbb{C}^* . In particular, they must have the same length, so $r = r'$. Continuing as in [BGM13, Theorem 4.3], we obtain that there is an isomorphism $\sigma : X \rightarrow X'$. Taking the pullback with respect to this isomorphism, Φ factorizes as

$$\begin{array}{ccc} \mathcal{M}(X, r, \xi) & \xrightarrow{\Phi} & \mathcal{M}(X', r, \xi') \\ & \searrow \bar{\Phi} & \nearrow (\sigma^{-1})^* \\ & \mathcal{M}(X, r, \sigma^* \xi') & \end{array}$$

and there is a commutative diagram

$$\begin{array}{ccc} T^* \mathcal{M}(X, r, \xi) & \xrightarrow{d\bar{\Phi}^{-1}} & T^* \mathcal{M}(X, r, \xi) \\ \mathcal{H} \downarrow & & \downarrow \mathcal{H} \\ \bigoplus_{j=2}^r H^0(X, K_X^j) & \xrightarrow{\bar{f}} & \bigoplus_{j=2}^r H^0(X, K_X^j) \end{array}$$

Then we can repeat the argument of [BGM13, Theorem 5.3] to obtain that $f = \text{Id}$ and if $E \in \mathcal{M}(X, r, \xi)$ is a generic vector bundle and $E' = \bar{\Phi}(E) \in \mathcal{M}(X, r, \sigma^* \xi')$ is its image, then

$$\text{End}_0(E) \cong \text{End}_0(E')$$

as Lie algebra bundles over X . Therefore, [BGM13, Lemma 5.4] applies and we obtain that there exists a line bundle L over X such that either $E' \cong E \otimes L$ or $E' \cong E^\vee \otimes L$. Taking degrees we obtain that $\deg(\sigma^* \xi') = \deg(\xi) + r \deg(L)$ or $\deg(\sigma^* \xi') = -\deg(\xi) + r \deg(L)$ respectively, so

$$\deg(\xi') = \deg(\sigma^* \xi') \cong \pm \deg(\xi) \pmod r$$

□

While, as we have showed, this result is a direct consequence of the work of Biswas, Gómez and Muñoz, we have not found this result explicitly stated in the literature, apart from the well known cases, such as the distinction between coprime and non-coprime cases or the explicitly computed isomorphism classes of the moduli

spaces for low genus curves and low rank. It is probable that there are other more simple topological techniques that allowed us distinguish the isomorphism classes of moduli spaces with determinants of different degree, but we are not aware of any method that allows us to distinguish all cases.

For example, in [BBGN07, p. 267, Theorem 1.8] it is proven that the Brauer group of the stable part of the moduli space $\text{Br}(\mathcal{M}^s(X, r, \xi))$ is isomorphic to $\mathbb{Z}/\delta\mathbb{Z}$, where $\delta = \text{g.c.d.}(r, \deg(\xi))$. If $\mathcal{M}(X, r, \xi) \cong \mathcal{M}(X, r, \xi')$, and it is generated by the class of the projectivized Poincaré bundle. Then we must have an isomorphism between their smooth (and therefore, stable) subsets $\mathcal{M}^s(X, r, \xi) \cong \mathcal{M}^s(X, r, \xi')$, so their Brauer groups must be isomorphic. In particular, the order of the Brauer group must be the same, and we obtain that

$$\text{g.c.d.}(r, \deg(\xi)) = \text{g.c.d.}(r, \deg(\xi'))$$

For example, this simple strategy allows us to distinguish all possible isomorphism classes of moduli spaces of rank $r \leq 4$, as for $r = 2$ and $r = 3$, it proves that

$$\begin{aligned} \mathcal{M}(X, 2, \mathcal{O}_X) &\not\cong \mathcal{M}(X, 2, \mathcal{O}_X(1)) \\ \mathcal{M}(X, 3, \mathcal{O}_X) &\not\cong \mathcal{M}(X, 3, \mathcal{O}_X(1)) \cong \mathcal{M}(X, 3, \mathcal{O}_X(2)) \end{aligned}$$

and for $r = 4$ it distinguishes $\mathcal{M}(X, 4, \mathcal{O}_X)$, $\mathcal{M}(X, 4, \mathcal{O}_X(1)) \cong \mathcal{M}(X, 4, \mathcal{O}_X(3))$ and $\mathcal{M}(X, 4, \mathcal{O}_X(2))$, as their respective sizes of their Brauer groups are 4, 1 and 2. Nevertheless, for $r = 5$ the Brauer group alone is not able to distinguish between the non-isomorphic moduli spaces $\mathcal{M}(X, 5, \mathcal{O}_X(1))$ and $\mathcal{M}(X, 5, \mathcal{O}_X(2))$, as it is trivial in both cases.

4.11.2 Wall crossings and k -birational classes

I would like to make some further comments on the evolution of the k -birational maps and their structure as we increase the codimension k and we consider curves of successively high genera g . As we mentioned earlier, the 1-birational class of all the moduli spaces $\mathcal{M}(X, r, \alpha, \xi)$ is exactly the same as far as we fix the genus of the curve X and the rank r for a full flag type, as they are all rational varieties of the same dimension [BY99]. On the other hand, in Theorem 4.7.5, we proved that if the genus of X and X' is at least 4, then two moduli spaces $\mathcal{M}(X, r, \alpha, \xi)$ and $\mathcal{M}(X', r', \alpha', \xi')$ are 3-birational if and only if

1. $(X, D) \cong (X', D')$
2. $r = r'$

Nevertheless, if we increase k , for any $k \geq 3$ one of the implications hold, namely, the “weak” version of the Torelli theorem. If $\mathcal{M}(X, r, \alpha, \xi) \stackrel{k\text{-bir}}{\cong} \mathcal{M}(X', r', \alpha', \xi')$ then $(X, D) \cong (X', D')$ and $r = r'$. Notice that our proof of the reciprocal, proved in Proposition 4.7.6, only works if $g \geq 1 + \frac{k-1}{r-1}$. and, in the extremal case $k = \dim(\mathcal{M}(X, r, \alpha, \xi))$, the previously stated refined Torelli Theorem (Theorem 4.11.1) implies that the reciprocal cannot be true. Observe that

$$\dim(\mathcal{M}(X, r, \alpha, \xi)) = (r^2 - 1)(g - 1) + n \frac{r^2 - r}{2}$$

while the maximum k for which the Proposition 4.7.6 holds is

$$k \leq (r-1)(g-1) + 1 \leq \frac{\dim(\mathcal{M}(X, r, \alpha, \xi))}{r+1} + 1$$

so as we continue to grow up k pass that point, we start fragmenting the k -birational equivalence classes in smaller subclasses that, once we reach the extremal case $k = \dim(\mathcal{M}(X, r, \alpha, \xi))$, are in correspondence with the orbits $\tilde{\Theta}/\mathcal{T}$ previously described.

We can also see this effect with the automorphisms of each moduli space $\mathcal{M}(X, r, \alpha, \xi)$. Corollary 4.7.11 tells us that

$$\text{Aut}_{3\text{-Bir}}(\mathcal{M}(r, \alpha, \xi)) = \mathcal{T}_\xi = \{T \in \mathcal{T} | T(\xi) \cong \xi\}$$

while Theorem 4.6.24 (re-written in terms of our classification space $\tilde{\Theta}/\mathcal{T}$, tells us that

$$\text{Aut}(\mathcal{M}(r, \alpha, \xi)) = \left\{ T \in \mathcal{T} \left| \begin{array}{c} T(\xi) \cong \xi \\ (\deg(\xi), T(\alpha)) \sim_{\Theta} (\deg(\xi), \alpha) \end{array} \right. \right\} = \mathcal{T}_\xi \cap \text{stab}_{\mathcal{T}}([(\deg(\xi), \alpha)])$$

where $\text{stab}_{\mathcal{T}}([(\deg(\xi), \alpha)])$ is the stabilizer of the class $[(\deg(\xi), \alpha)] \in \tilde{\Theta}$ for the action of \mathcal{T} . The difference between both scenarios is that for a basic transformation $T \in \mathcal{T}_\xi$ to induce an automorphism, the target stability $T(\alpha)$ and the origin stability α must belong to the same stability chamber, thus identifying the moduli spaces $\mathcal{M}(X, r, \alpha, \xi)$ and $\mathcal{M}(X, r, T(\alpha), \xi)$. If they do not belong to the same chamber, we can find a parabolic vector bundle (E, E_\bullet) on X with determinant ξ which is α -stable, but not $T(\alpha)$ -stable and we proved that this is enough to prevent T from induce an isomorphism. On the other hand, from Corollary 4.1.4 we know that the set of α -stable and $T(\alpha)$ -stable is “small” in the sense that it has codimension at least 3 (for $g \geq 4$). Therefore, out of that “small” set, the moduli spaces can indeed be identified. The cost is that the induced map by T is not anymore an isomorphism, but just a 3-birational equivalence.

In general, we observe the following principle. Let us fix a line bundle ξ of degree d . Let $T \in \mathcal{T}_\xi$ be a basic transformation preserving some determinant ξ . Let us fix a starting chamber α and let us consider the chamber $T(\alpha)$ reachable by $T \in \mathcal{T}_\xi$. We have two possibilities

1. α and $T(\alpha)$ belong to the same numerical stability chamber. In other words, $\overline{M}(r, \alpha, d) = \overline{M}(r, T(\alpha), d)$. In this case we know for sure that T induces an automorphism of $\mathcal{M}(X, r, \alpha, \xi)$.
2. α and $T(\alpha)$ do not belong to the same numerical stability chamber. Then they might not belong to the same geometrical stability chamber, so, a priori, we know that T induces an isomorphism $T : \mathcal{M}(X, r, \alpha, \xi) \rightarrow \mathcal{M}(X, r, T(\alpha), \xi)$, but we do not know if we can identify the target and origin moduli spaces to obtain an isomorphism.

Treating case 2 can be really hard, as it involves determining which numerical barriers in the stability space can actually be realized by geometric examples, i.e., for which barriers there exists a weight α_{lim} for which strictly α_{lim} -semistable parabolic vector bundles exist. This usually involves some Brill-Noether theory and a complete description depends heavily on the geometry of the curve.

On the other hand, if we broaden our point of view, we might step away from the binary classification of numerical barriers into just numerical (there does not exist a strictly semistable parabolic vector bundle in the barrier) and geometrical (having a parabolic bundle realizing the barrier). Instead, we can classify the barriers according to the codimension of the space of realizations of the barrier.

For each $\beta \in \tilde{\Delta}$, let $\mathcal{Z}_{\alpha \rightarrow \beta} \subset \mathcal{M}(X, r, \alpha, \xi)$ be the subset of parabolic vector bundles (E, E_\bullet) of rank r and determinant ξ which are α -stable but not β -stable. If $\mathcal{Z}_{\alpha \rightarrow \beta} = \emptyset$, then α and β belong to the same stability chamber. In other case, the elements in $\mathcal{Z}_{\alpha \rightarrow \beta}$ explicitly realize at least one numerical barrier separating the numerical chambers of α and β . Nevertheless, if our aim is to compare the moduli spaces $\mathcal{M}(X, r, \alpha, \xi)$ and $\mathcal{M}(X, r, \beta, \xi)$, then the actual size (in the sense of dimension in this case) of the set $\mathcal{Z}_{\alpha \rightarrow \beta}$ matters. Observe that there is a canonical identification

$$\mathcal{M}(X, r, \alpha, \xi) \setminus \mathcal{Z}_{\alpha \rightarrow \beta} \cong \mathcal{M}(X, r, \beta, \xi) \setminus \mathcal{Z}_{\beta \rightarrow \alpha}$$

and the sets $\mathcal{Z}_{\alpha \rightarrow \beta}$ and $\mathcal{Z}_{\beta \rightarrow \alpha}$ are closed, meaning that both moduli spaces are always birational. The smaller the sets $\mathcal{Z}_{\alpha \rightarrow \beta}$ and $\mathcal{Z}_{\beta \rightarrow \alpha}$ are, the more similar the moduli spaces are. More precisely, if

$$k = \max\{\text{codim}(\mathcal{Z}_{\alpha \rightarrow \beta}), \text{codim}(\mathcal{Z}_{\beta \rightarrow \alpha})\}$$

then the moduli spaces are k -birational. We have given several examples showing the importance of controlling k -birationality for higher order k , as it implies that more and more invariants are shared between the moduli spaces. In particular, this is important for understanding the behavior of our basic transformations. In our previous example, composing the map $T : \mathcal{M}(X, r, \alpha, \xi) \rightarrow \mathcal{M}(X, r, T(\alpha), \xi)$ with the correspondence $\mathcal{M}(X, r, T(\alpha), \xi) \dashrightarrow \mathcal{M}(X, r, \alpha, \xi)$ induces canonically a k -birational equivalence (that we also denote by T), $T : \mathcal{M}(X, r, \alpha, \xi) \dashrightarrow \mathcal{M}(X, r, \alpha, \xi)$, where

$$k = \max\{\text{codim}(\mathcal{Z}_{\alpha \rightarrow T(\alpha)}), \text{codim}(\mathcal{Z}_{T(\alpha) \rightarrow \alpha})\}$$

Back to our results, Corollary 4.1.4 can be rephrased saying that for each curve of genus g and for each generic α and β

$$\text{codim}(\mathcal{Z}_{\alpha \rightarrow \beta}) \geq (r-1)(g-1) + 1$$

On the other hand, Theorem 4.9.6 implies that for each α and β and each curve X of genus

$$g \geq 1 + (r-1)n - \min\left(\left|\sum_{x \in D} \alpha_1(x)\right|, \left|\sum_{x \in D} \beta_1(x)\right|\right)$$

then $\mathcal{Z}_{\alpha \rightarrow \beta} \neq \emptyset$. Moreover, for certain choices of α and β this bound can be significantly improved, as shown in Proposition 4.9.7 and the comments made after the proposition.

In other words, if we fix two systems α and β , while computing $\mathcal{Z}_{\alpha \rightarrow \beta}$ might be challenging, we can actually find bounds for its dimension. In particular, for our basic transformation T , while it is not straightforward to compute the exact k for which $T : \mathcal{M}(X, r, \alpha, \xi) \dashrightarrow \mathcal{M}(X, r, \alpha, \xi)$ is a k -birational equivalence, we can give bounds for it.

We can think of the following picture. Fix a system of weights α of rank r over n points. Consider a curve X of genus g and a basic transformation $T \in \mathcal{T}$ sending α to $T(\alpha)$ with α and $T(\alpha)$ in different numerical chambers. The parabolic system of weights $T(\alpha)$ does not really depend on the choice of T or X , just on how it permutes the parabolic points and weights, so we may consider basic transformations T_X for different curves X inducing the same change in the weights. Then we might do so while increasing the genus of the curve X and observe the subsequent changes in the regularity of the birational equivalence $T_X : \mathcal{M}(X, r, \alpha, \xi) \dashrightarrow \mathcal{M}(X, r, \alpha, \xi)$.

1. If we start with **low genus** $g \geq 4$, we know that T_X is at least a 3-birational map. Moreover, depending on the geometry of the curve, the choice of α and other factors such as the choice of the determinant, it is possible that the barriers between α and $T(\alpha)$ are not realized by any actual strictly semistable parabolic vector bundle and, in that case, T_X would be an automorphism.
2. As we **grow the genus**, we also grow linearly the minimum regularity of the birational map. In particular, we know that T_X is at least $((r-1)(g-1)+1)$ -birational. If g is not very high, then the barriers between α and $T(\alpha)$ **might not be geometrical**, so T_X might be an automorphism. We know for sure that there will be a genus g from which the chambers α and $T(\alpha)$ are geometrically separated and, for certain chambers, this genus can be really low. For example, if α is in the concentrated chamber and T_X acts on the weights as a Hecke transform, Theorem 4.8.2 proves that for $g \geq 6$, then α and $T(\alpha)$ are already separated by a geometrical barrier, so T_X cannot be an isomorphism.
3. On the other hand, we know for sure that for certain that if g **exceeds the bound** given by Proposition 4.9.7, then α and $T_X(\alpha)$ are **not in the same stability chamber** for any curve X or choice of the transformation T_X , so T_X **is never an automorphism**. Moreover, for high bounds of the genus, we can compute exactly the codimension of $\mathcal{Z}_{\alpha \rightarrow T(\alpha)}$ working as in Lemma 4.9.5, but instead of just using [BB05, Theorem 1.4.3A] to prove the non-emptiness of the corresponding Segre strata of the moduli space constituting $\mathcal{Z}_{\alpha \rightarrow T(\alpha)}$, we combine it with the dimension computation of the nonempty strata given in [BB05, Theorem 1.4.1]. As $\mathcal{Z}_{\alpha \rightarrow T(\alpha)}$ is formed by a finite union of such strata, the dimension of $\mathcal{Z}_{\alpha \rightarrow T(\alpha)}$ is just the maximum of the dimensions given by [BB05, Theorem 1.4.1] for the suitable choices of the rank of the subbundle r' , the Segre invariant s and the parabolic type \bar{n} determined in the proof of Theorem 4.9.6.

4.11.3 Torelli theorems for the moduli space of parabolic vector bundles with fixed degree

I want to make a remark concerning the choice of working with the moduli space of parabolic vector bundles with fixed determinant, instead of working with the moduli space of parabolic vector bundles with fixed degree.

In our current proof the fact that the determinant is fixed plays a key role in several parts of the proof. On one hand, it implies that the Higgs bundles arising

as part of the cotangent bundle of the moduli space $\mathcal{M}(X, r, \alpha, \xi)$ are traceless and, therefore, the associated Hitchin map lacks the trace factor $H^0(K)$. This turns out to be really important when analyzing the geometry of the Hitchin space, as the lack of a degree one block for the canonical \mathbb{C}^* -action on the Hitchin space W implies the non-existence of certain \mathbb{C}^* -equivariant automorphisms of W .

On the other hand, fixing the determinant imposes hard restrictions on the possible candidates for an automorphism of the moduli space $\mathcal{M}(X, r, \alpha, \xi)$. Once we have proven that the possible automorphisms behave pointwise as basic transformations, fixing the determinant implies that the number of possible such transformations is actually finite, leading us to proving that there must be, in fact, a single basic transformation acting globally on the moduli space and inducing the corresponding automorphism.

Nevertheless, we will prove that our results on the fixed determinant moduli space in fact imply analogous Torelli type theorems for the moduli space of parabolic vector bundles with fixed degree.

We will start by using a classical strategy to recover the determinant map from the moduli space of fixed degree.

Lemma 4.11.3. *Let α be a full flag system of weights. Then the determinant map $\mathcal{M}(X, r, \alpha, d) \rightarrow \text{Pic}^d(X)$ sending (E, E_\bullet) to $\det(E)$ is the Albanese map for $\mathcal{M}(X, r, \alpha, d)$.*

Proof. If α is full flag, then by [BY99], for every $\xi \in \text{Pic}^d(X)$, the moduli space $\mathcal{M}(X, r, \alpha, \xi)$ is a rational variety. Let \mathcal{A} be any abelian variety admitting a map $f : \mathcal{M}(X, r, \alpha, d) \rightarrow \mathcal{A}$. Then, for any $\xi \in \text{Pic}^d(X)$ we obtain a map $\det^{-1}(\xi) = \mathcal{M}(X, r, \alpha, \xi) \rightarrow \mathcal{A}$. As $\mathcal{M}(X, r, \alpha, \xi)$ is rational and \mathcal{A} is abelian, then the map must be constant. Therefore, the map $f : \mathcal{M}(X, r, \alpha, d) \rightarrow \mathcal{A}$ is constant in the fibers of the map $\det : \mathcal{M}(X, r, \alpha, d) \rightarrow \text{Pic}^d(X)$, so it descends to a map $\text{Pic}^d(X) \rightarrow \mathcal{A}$

$$\begin{array}{ccc} \mathcal{M}(X, r, \alpha, \xi) & \xrightarrow{f} & \mathcal{A} \\ \downarrow \det & \nearrow & \\ \text{Pic}^d(X) & & \end{array}$$

By the universal property of the Albanese map, we conclude that $\det : \mathcal{M}(X, r, \alpha, d) \rightarrow \text{Pic}^d(X)$ is isomorphic to the Albanese map. \square

Now, we can use this result to obtain a Torelli type theorem for the whole space using the Torelli for its fibers.

Theorem 4.11.4. *Let (X, D) and (X', D') be two smooth projective curves of genus $g \geq 6$ and $g' \geq 6$ respectively with set of marked points $D \subset X$ and $D' \subset X'$. Let α and α' be full flag generic systems of weights over (X, D) and (X', D') respectively. Then $\mathcal{M}(X, r, \alpha, d) \cong \mathcal{M}(X', r', \alpha', d')$ if and only if*

1. $r = r'$
2. (X, D) is isomorphic to (X', D') , i.e., there exists an isomorphism $\sigma : X \xrightarrow{\sim} X'$ sending D to D' .

3. In $\tilde{\Theta}/\mathcal{T}$ we have $(d, \alpha) \sim_{\Theta} (d', \sigma^* \alpha')$

Proof. Let $\Phi : \mathcal{M}(X, r, \alpha, d) \rightarrow \mathcal{M}(X', r', \alpha', d')$ be an isomorphism. The previous lemma tells us this map induces an isomorphism of the corresponding Albanese varieties, so there is a map $\varphi : \text{Pic}^d(X) \rightarrow \text{Pic}^{d'}(X')$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{M}(X, r, \alpha, d) & \xrightarrow{\Phi} & \mathcal{M}(X', r', \alpha', d') \\ \det \downarrow & & \downarrow \det \\ \text{Pic}^d(X) & \xrightarrow{\varphi} & \text{Pic}^{d'}(X') \end{array}$$

Pick any $\xi \in \text{Pic}^d(X)$ and let $\xi' = \varphi(\xi)$. Then Φ induces an isomorphism

$$\Phi_{\psi} : \mathcal{M}(X, r, \alpha, \xi) \xrightarrow{\det^{-1}} \mathcal{M}(X, r, \alpha, \xi) \xrightarrow{\det^{-1}} \mathcal{M}(X', r', \alpha', \xi')$$

Now, applying Theorem 4.11.1 we obtain that

1. $r = r'$
2. (X, D) is isomorphic to (X', D')
3. In $\tilde{\Theta}/\mathcal{T}$ we have $(d, \alpha) \sim_{\Theta} (d', \sigma^* \alpha')$

On the other hand, suppose that (1), (2) and (3) hold. Taking the pullback with respect to the isomorphism $\sigma : X \rightarrow X'$ we might assume that $(X, D) = (X', D')$. As $(d, \alpha) \sim_{\Theta} (d', \sigma^* \alpha')$ then by definition of the relation \sim_{Θ} there exists a basic transformation T sending α -stable parabolic vector bundles of degree d to α' -stable parabolic vector bundles of degree d' . Moreover, by its construction (see Section 4.4), it is straightforward to check that the induced map $T : \text{Pic}^d(X) \rightarrow \text{Pic}^{d'}(X)$ is an isomorphism, so T induces an isomorphism $T : \mathcal{M}(X, r, \alpha, d) \rightarrow \mathcal{M}(X, r, \alpha', d')$. \square

This result represents a twofold generalization of the Torelli theorem developed by Biswas, Gómez and Logares [BGL16] in the case of full flag systems of weights. On one hand, it proves that the isomorphism class can be recovered directly from the isomorphism class of the moduli space, without the need of an additional polarization. On the other hand, it provides a reciprocal for the theorem, in the form of a necessary and sufficient condition for two moduli spaces of parabolic vector bundles with fixed degree to be isomorphic.

Nonetheless, I would like to remark that the previous proof is intrinsically restricted to the full flag case or, at least, to systems of weights having at least a jump of order one, as we need the rationality of the moduli space to prove that the isomorphism $\mathcal{M}(X, r, \alpha, d) \rightarrow \mathcal{M}(X', r', \alpha', d')$ respects the fibers of the determinant map. In general, for other parabolic types the moduli space might not be rational and other kind of results would be necessary to prove that the Albanese of $\mathcal{M}(X, r, \alpha, \xi)$ is a point. It may be still possible that for an arbitrary parabolic type it is necessary to know the determinantal polarization of the moduli space in order to recover the isomorphism class of the curve.

Moreover, we can repeat the previous argument for a k -birational map instead of an isomorphism.

Lemma 4.11.5. *Let α and α' be full flag systems of weights over marked curves (X, D) and (X', D') respectively. Let $\Phi : \mathcal{M}(X, r, \alpha, d) \dashrightarrow \mathcal{M}(X', r', \alpha', d')$ be a birational map. Then there is an isomorphism $\varphi : \text{Pic}^d(X) \dashrightarrow \text{Pic}^{d'}(X')$ such that the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{M}(X, r, \alpha, d) & \xrightarrow{\quad \Phi \quad} & \mathcal{M}(X', r', \alpha', d') \\ \det \downarrow & & \downarrow \det \\ \text{Pic}^d(X) & \xrightarrow{\quad \varphi \quad} & \text{Pic}^{d'}(X') \end{array}$$

Proof. We proceed as in Lemma 4.11.3, but restricting ourselves to the open subset where Φ is well defined. Let $\mathcal{U} \subset \mathcal{M}(X, r, \alpha, d)$ and $\mathcal{U}' \subset \mathcal{M}(X', r', \alpha', d')$ be open subsets such that $\Phi : \mathcal{U} \rightarrow \mathcal{U}'$ is an isomorphism. Let \mathcal{Z} and \mathcal{Z}' be the complements of \mathcal{U} and \mathcal{U}' respectively. As $\dim(\mathcal{Z}) \leq \dim(\mathcal{M}(X, r, \alpha, d)) - 1$, the generic fiber of the map $\det : \mathcal{Z} \rightarrow \text{Pic}^d(X)$ must have dimension at most $\dim(\mathcal{M}(X, r, \alpha, d)) - \dim(\text{Pic}^d(X)) - 1 = \dim(\mathcal{M}(X, r, \alpha, \xi)) - 1$. The dimension of the fiber is upper semicontinuous, so there is an open dense subset $\mathcal{P} \subset \text{Pic}^d(X)$ such that for every $\xi \in \mathcal{P}$, $\dim(\mathcal{Z} \cap \det^{-1}(\xi)) \leq \dim(\det^{-1}(\xi)) - 1$. Therefore, for each $\xi \in \mathcal{P}$, $\det^{-1}(\xi) \cap \mathcal{U}$ is dense in $\mathcal{M}(X, r, \alpha, \xi)$ and, therefore, $\det^{-1}(\xi) \cap \mathcal{U}$ is rational.

Let $\tilde{\mathcal{U}} = \det^{-1}(\mathcal{P}) \cap \mathcal{U}$. It is an open dense subset of $\mathcal{M}(X, r, \alpha, d)$ for which Φ is well defined. Therefore, by composition, it admits a map $\tilde{\mathcal{U}} \rightarrow \mathcal{U}' \rightarrow \text{Pic}^{d'}(X')$. As the fibers of $\tilde{\mathcal{U}}$ for the determinant map are rational, each fiber must map to a single point in $\text{Pic}^{d'}(X')$, so this map descends to a map $\varphi : \mathcal{P} \rightarrow \text{Pic}^{d'}(X')$. As the map $\det : \mathcal{M}(X', r', \alpha', d') \rightarrow \text{Pic}^{d'}(X')$ is surjective with equidimensional fibers and $\Phi(\tilde{\mathcal{U}}) \subset \mathcal{U}'$ is dense, then its image $\varphi(\mathcal{P})$ is dense in $\text{Pic}^{d'}(X')$. Repeating the argument for Φ^{-1} proves that there are open dense subsets $\tilde{\mathcal{P}} \subset \text{Pic}^d(X)$ and $\tilde{\mathcal{P}}' \subset \text{Pic}^{d'}(X')$ such that $\varphi : \tilde{\mathcal{P}} \rightarrow \tilde{\mathcal{P}}'$ is an isomorphism, thus inducing a birational map $\text{Pic}^d(X) \dashrightarrow \text{Pic}^{d'}(X')$. Birational maps between abelian varieties extend uniquely to isomorphisms (c.f. [Mil08, Theorem 3.8]), so φ extends to an isomorphism $\varphi : \text{Pic}^d(X) \rightarrow \text{Pic}^{d'}(X')$. \square

Theorem 4.11.6. *Let (X, D) and (X', D') be two smooth projective curves of genus $g \geq 4$ and $g' \geq 4$ respectively with set of marked points $D \subset X$ and $D' \subset X'$. Let α and α' be full flag generic systems of weights over (X, D) and (X', D') respectively. Then $\mathcal{M}(X, r, \alpha, d)$ and $\mathcal{M}(X', r', \alpha', d')$ are 3-birational if and only if*

1. $r = r'$
2. (X, D) is isomorphic to (X', D') , i.e., there exists an isomorphism $\sigma : X \xrightarrow{\sim} X'$ sending D to D' .

Proof. By the previous Lemma we know that $\dim(\text{Pic}^d(X)) = \dim(\text{Pic}^{d'}(X'))$ and $\dim(\mathcal{M}(X, r, \alpha, d)) = \dim(\mathcal{M}(X', r', \alpha', d'))$ so, as the determinant map is equidimensional, we know that the dimension of their respective fibers is the same.

As before, let $\mathcal{U} \subset \mathcal{M}(X, r, \alpha, d)$ and $\mathcal{U}' \subset \mathcal{M}(X', r', \alpha', d')$ be open subsets such that there is an isomorphism $\Phi : \mathcal{U} \rightarrow \mathcal{U}'$. Let \mathcal{Z} and \mathcal{Z}' be their respective complements in the moduli spaces. We know that $\text{codim}(\mathcal{Z}) \geq 3$, and the map $\det : \mathcal{M}(X, r, \alpha, d) \rightarrow \text{Pic}^d(X)$ is equidimensional and surjective, so for a generic

$\xi \in \text{Pic}^d(X)$, the codimension of $\mathcal{Z} \cap \det^{-1}(\xi)$ in $\mathcal{M}(X, r, \alpha, \xi)$ is at least 3. Similarly, for a generic $\xi' \in \text{Pic}^{d'}(X')$, the codimension of $\mathcal{Z} \cap \det^{-1}(\xi')$ in $\mathcal{M}(X', r', \alpha', \xi')$ is at least 3. The map $\varphi : \text{Pic}^d(X) \rightarrow \text{Pic}^{d'}(X')$ is an isomorphism, so there exists $\xi \in \text{Pic}^d(X)$ and $\xi' = \varphi(\xi) \in \text{Pic}^{d'}(X')$ such that

$$\Phi(\mathcal{M}(X, r, \alpha, \xi) \cap \mathcal{U}) = \mathcal{M}(X', r', \alpha', \xi') \cap \mathcal{U}'$$

$$\text{codim}(\mathcal{Z} \cap \mathcal{M}(X, r, \alpha, \xi)) \geq 3$$

$$\text{codim}(\mathcal{Z}' \cap \mathcal{M}(X', r', \alpha', \xi')) \geq 3$$

Thus, Φ induces a 3-birational equivalence between $\mathcal{M}(X, r, \alpha, \xi)$ and $\mathcal{M}(X', r', \alpha', \xi')$. Now we can apply the k -birational version of the Torelli theorem, Theorem 4.7.5. The reciprocal is proven exactly as in Proposition 4.7.6, but applying Lemma 4.1.3 directly instead of Corollary 4.1.4. \square

Chapter 5

Automorphism group of a moduli space of framed bundles over a curve

The contents of this chapter have been developed in collaboration with Indranil Biswas and can be found in [AB18].

Framed bundles (also called vector bundles with a level structure) are pairs (E, α) consisting of a vector bundle E of rank r and a nonzero linear map $\alpha : E_x \rightarrow \mathbb{C}^r$ from a fiber over a fixed point $x \in X$ to \mathbb{C}^r ; this α is called a framing. Framed bundles were first introduced by Donaldson as a tool to study the moduli space of instantons on \mathbb{R}^4 [Don84]. Latter on, Huybrechts and Lehn [HL95a, HL95b] defined framed modules as a common generalization of several notions of decorated sheaves including framed bundles and Bradlow pairs. They described a general stability condition for framed modules and provided a GIT construction for the moduli space of framed modules.

A moduli space of framed bundles of rank r carries a canonical $\mathrm{PGL}_r(\mathbb{C})$ -action that sends each $[G] \in \mathrm{PGL}_r(\mathbb{C})$ and each framed bundle (E, α) to

$$[G] \cdot (E, \alpha) = (E, G \circ \alpha).$$

In [BGM10], a Torelli type theorem was proved for the moduli space of framed bundles by studying this $\mathrm{PGL}_r(\mathbb{C})$ -action. It was proved there that this action is essentially the only nontrivial $\mathrm{PGL}_r(\mathbb{C})$ -action on the moduli space; the corresponding GIT-quotient was shown to be isomorphic to the moduli space of vector bundles.

Our aim here is to compute the automorphism group of the moduli space of framed bundles with fixed determinant; towards this the following is proved (see Theorem 5.3.6):

Theorem 5.0.1. *Let X be a smooth complex projective curve of genus $g > 2$ with a base point x . If τ is a small stability parameter, then the automorphism group of the moduli space of τ -semistable framed bundles with fixed determinant ξ and framing over x is generated by the following transformations*

- *pullback with respect to the automorphisms $\sigma : X \rightarrow X$ that fix the point $x \in X$,*

- tensorization with a line bundle $L \in \text{Pic}(X)$, and
- action of $\text{PGL}_r(\mathbb{C})$ defined by $[G] \cdot (E, \alpha) = (E, G \circ \alpha)$,

where σ and L satisfy the relation $\sigma^* \xi \otimes L^{\otimes r} \cong \xi$.

In particular, this allows us to compute explicitly the structure of the automorphism group of the moduli space of framed bundles \mathcal{F} (Corollary 5.3.7):

Corollary 5.0.2. *The automorphism group of \mathcal{F} is*

$$\text{Aut}(\mathcal{F}) \cong \text{PGL}_r(\mathbb{C}) \times \mathcal{T}$$

for a group \mathcal{T} fitting in the short exact sequence

$$1 \longrightarrow J(X)[r] \longrightarrow \mathcal{T} \longrightarrow \text{Aut}(X, x) \longrightarrow 1,$$

where $J(X)[r]$ is the r -torsion part of the Jacobian of X and

$$\text{Aut}(X, x) = \{\sigma \in \text{Aut}(X) \mid \sigma(x) = x\}.$$

The classification of the automorphisms of the moduli space of vector bundles carried out in [KP95] plays an important role in the computations done in Theorem 5.0.1 and Corollary 5.0.2.

5.1 Moduli space of framed bundles

Let X be a smooth complex projective curve. Fix a point $x \in X$. A framed bundle on (X, x) is a pair (E, α) consisting on a vector bundle E over X and a nonzero \mathbb{C} -linear homomorphism

$$\alpha : E_x \longrightarrow \mathbb{C}^r.$$

Given a real number $\tau > 0$, we say that a framed bundle (E, α) is τ -stable (respectively τ -semistable) if for all proper subbundles $0 \subsetneq E' \subsetneq E$

$$\frac{\text{degree}(E') - \epsilon(E', \alpha)\tau}{\text{rk}(E')} < \frac{\text{degree}(E) - \tau}{\text{rk}(E)} \quad (\text{respectively, } \leq)$$

where

$$\epsilon(E', \alpha) = \begin{cases} 1 & \text{if } E'_x \not\subseteq \ker(\alpha) \\ 0 & \text{if } E'_x \subseteq \ker(\alpha). \end{cases}$$

In the general framework of framed modules introduced in [HL95a], a framed bundle is a framed module with respect to the reference sheaf $\mathcal{O}_x^{\oplus r}$. The stability condition for framed bundles described here coincides with the stability condition defined by Huybrechts and Lehn for framed modules. Fix a line bundle ξ on X . Let $\mathcal{F} = \mathcal{F}(X, x, r, \xi, \tau)$ be the moduli space of τ -semistable framed bundles (E, α) on (X, x) with $\text{rank}(E) = r$ and $\det(E) = \bigwedge^r E \cong \xi$. By [HL95a], it is a complex projective variety.

On the other hand, a vector bundle E is called stable (respectively semistable) if for all proper subbundles $0 \subsetneq E' \subsetneq E$

$$\frac{\text{degree}(E')}{\text{rk}(E')} < \frac{\text{degree}(E)}{\text{rk}(E)} \quad (\text{respectively, } \leq)$$

Let $\mathcal{M} = \mathcal{M}(X, r, \xi)$ denote the moduli space of semistable vector bundles over X of rank r and determinant ξ .

By [BGM10, Lemma 1.1], there exists some constant $\tau_0(r)$ depending only on the rank r such that if $0 < \tau < \tau_0(r)$ then the following implications hold

$$E \text{ stable} \implies (E, \alpha) \text{ } \tau\text{-stable} \iff (E, \alpha) \text{ } \tau\text{-semistable} \implies E \text{ semistable}.$$

From now on, we assume that $0 < \tau < \tau_0(r)$. Then there is a forgetful map

$$\begin{aligned} f : \mathcal{F} &\longrightarrow \mathcal{M} \\ (E, \alpha) &\longmapsto E \end{aligned}$$

We can make $\mathrm{PGL}_r(\mathbb{C})$ act on \mathcal{F} by composition with the framing α . Given a matrix $[G] \in \mathrm{PGL}_r(\mathbb{C})$, where $G \in \mathrm{GL}_r(\mathbb{C})$ is any representative of the projective class, the automorphism $G : \mathbb{C}^r \longrightarrow \mathbb{C}^r$ produces the self-map

$$(E, \alpha) \longmapsto (E, G \circ \alpha)$$

of framed bundles. Since for every subbundle $E' \subset E$ we have

$$\epsilon(E', \alpha) = \epsilon(E', G \circ \alpha)$$

this transformation preserves the (semi)stability condition and it gives a well defined map $\varphi_G : \mathcal{F} \longrightarrow \mathcal{F}$.

On the other hand, we can perform the following transformations on (families of) framed bundles (E, α) which preserve the stability condition:

1. Given an automorphism $\sigma : X \longrightarrow X$ that fixes $x \in X$,

$$(E, \alpha) \longmapsto (\sigma^* E, \alpha).$$

2. Given a line bundle L over X , fix a trivialization $\alpha_L : L_x \xrightarrow{\sim} \mathbb{C}$. Then send

$$(E, \alpha) \longmapsto (E \otimes L, \alpha \cdot \alpha_L)$$

Since two trivializations α_L and α'_L differ only by a scalar constant, this map is well defined and furthermore it is independent on the choice of the trivialization α_L .

Note that taking the pullback by σ and tensoring with L both change the determinant of the resulting framed bundle. Therefore, these transformations do not in general induce automorphism of the moduli space \mathcal{F} , but rather an isomorphism between $\mathcal{F}(X, x, r, \xi, \tau)$ and another moduli space of framed bundles with a different determinant $\mathcal{F}(X, x, r, \sigma^* \xi \otimes L^{\otimes r}, \tau)$. Nevertheless, if σ and L satisfy the relation $\sigma^* \xi \otimes L^{\otimes r} \cong \xi$, it is clear that the map $\overline{\mathcal{T}_{\sigma, L, +}} : \mathcal{F} \longrightarrow \mathcal{F}$ sending

$$(E, \alpha) \longmapsto \overline{\mathcal{T}_{\sigma, L, +}}(E, \alpha) = (\sigma^* E \otimes L, \alpha \cdot \alpha_L) \tag{5.1.1}$$

is an automorphism of the moduli space.

5.2 Framed bundles with invertible framing

Let \mathcal{F}^{ss} denote the subset of \mathcal{F} corresponding to framed bundles (E, α) such that α is an isomorphism; it is evidently Zariski open. Analogously, let \mathcal{F}^0 be the subset of \mathcal{F}^{ss} consisting on framed bundles (E, α) such that α is an isomorphism and E is a stable vector bundle; from the openness of the stability condition, [Mar81, p. 635, Theorem 2.8(B)], it follows that \mathcal{F}^0 is also Zariski open. As the action of $\mathrm{PGL}_r(\mathbb{C})$ on framed bundles preserves stability, and acts freely and transitively on the space of isomorphisms $E_x \xrightarrow{\sim} \mathbb{C}^r$, the fiber of the restricted forgetful map

$$f^0 : \mathcal{F}^0 \longrightarrow \mathcal{M}^s$$

over a stable vector bundle $E \in f^0(\mathcal{F}^0)$ is

$$(f^0)^{-1}(E) = \mathbb{P}(\mathrm{Isom}(E_x, \mathbb{C}^r)) \cong \mathrm{PGL}_r(\mathbb{C}).$$

Moreover, the map $f^0 : \mathcal{F}^0 \longrightarrow \mathcal{M}^s$ is surjective as a consequence of the following proposition.

Proposition 5.2.1. *If $\alpha : E|_x \longrightarrow \mathbb{C}^r$ is an isomorphism, then (E, α) is τ -stable if and only if E is semistable*

Proof. By [BGM10, Lemma 1.1], if (E, α) is τ -stable, then E is semistable. On the other hand, if α is an isomorphism, then for every subbundle $E' \subsetneq E$, we have $\alpha|_{E'} \not\cong 0$, so $\epsilon(E', \alpha) = 1$. Now as $\mathrm{rk}(E') < \mathrm{rk}(E)$ and $\tau > 0$, we have

$$\frac{-\epsilon(E', \alpha)\tau}{\mathrm{rk}(E')} = \frac{-\tau}{\mathrm{rk}(E')} < \frac{-\tau}{\mathrm{rk}(E)}$$

so if E is semistable, then for every $E' \subsetneq E$, we have

$$\frac{\mathrm{degree}(E')}{\mathrm{rk}(E')} - \frac{-\epsilon(E', \alpha)\tau}{\mathrm{rk}(E')} < \frac{\mathrm{degree}(E)}{\mathrm{rk}(E)} - \frac{\tau}{\mathrm{rk}(E)}.$$

□

On \mathcal{F}^0 we can define an additional transformation inducing an isomorphism

$$\mathcal{D} : \mathcal{F}^0(X, x, r, \xi, \tau) \xrightarrow{\sim} \mathcal{F}^0(X, x, r, \xi^{-1}, \tau)$$

in the following way. An isomorphism $\alpha : E_x \longrightarrow \mathbb{C}^r$ induces an isomorphism $\alpha^{-1} : \mathbb{C}^r \longrightarrow E_x$. Identifying $(\mathbb{C}^r)^\vee \cong \mathbb{C}^r$ and taking duals, we obtain an isomorphism $(\alpha^{-1})^t : E_x^\vee \longrightarrow \mathbb{C}^r$. Now take

$$\mathcal{D}(E, \alpha) = (E^\vee, (\alpha^{-1})^t).$$

Since the transformation is evidently well defined for families, to show that \mathcal{D} induces an isomorphism between the moduli spaces it is enough to prove that it preserves τ -semistability.

Proposition 5.2.2. *The framed bundle $\mathcal{D}(E, \alpha)$ is τ -semistable if and only if (E, α) is τ -semistable.*

Proof. Recall that the choice of τ implies that τ -semistability is equivalent to τ -stability. Therefore, by Proposition 5.2.1, the framed bundle $\mathcal{D}(E, \alpha)$ is τ -semistable if and only if E^\vee is semistable, while (E, α) is τ -semistable if and only if E is semistable. As E is semistable if and only if E^\vee is semistable, the result follows. \square

Let L be a line bundle over X , and let $\sigma : X \rightarrow X$ be an automorphism of the curve; take any $s \in \{1, -1\}$. We define the map $\mathcal{T}_{\sigma, L, s} : \mathcal{M}(X, r, \xi) \rightarrow \mathcal{M}(X, r, \sigma^* \xi^s \otimes L^{\otimes r})$ as

$$\begin{array}{ccc} \mathcal{T}_{\sigma, L, +} : \mathcal{M}(X, r, \xi) & \longrightarrow & \mathcal{M}(X, r, \sigma^* \xi \otimes L^{\otimes r}) \\ E \longmapsto & & \sigma^* E \otimes L \end{array}$$

for $s = 1$, and

$$\begin{array}{ccc} \mathcal{T}_{\sigma, L, -} : \mathcal{M}(X, r, \xi) & \longrightarrow & \mathcal{M}(X, r, \sigma^* \xi^{-1} \otimes L^{\otimes r}) \\ E \longmapsto & & \sigma^* E^\vee \otimes L \end{array}$$

for $s = -1$. If $\sigma^* \xi^s \otimes L^{\otimes r} \cong \xi$, then the above defined map $\mathcal{T}_{\sigma, L, s} : \mathcal{M} \rightarrow \mathcal{M}$ is an automorphism of the moduli space of vector bundles such that $\mathcal{T}_{\sigma, L, s}(\mathcal{M}^s) = \mathcal{M}^s$. In fact, by [KP95] and [BGM13], every automorphism of \mathcal{M} is given by a transformation of type $\mathcal{T}_{\sigma, L, s}$. Analogously, if L is a line bundle over X with $\sigma : X \rightarrow X$ an automorphism fixing $x \in X$, and $s \in \{1, -1\}$, then define

$$\begin{array}{ccc} \mathcal{T}_{\sigma, L, +}^0 : \mathcal{F}^0(X, x, r, \xi, \tau) & \longrightarrow & \mathcal{F}^0(X, x, r, \sigma^* \xi \otimes L^{\otimes r}, \tau) \\ (E, \alpha) \longmapsto & & (\sigma^* E \otimes L, \alpha \cdot \alpha_L) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{T}_{\sigma, L, -}^0 : \mathcal{F}^0(X, x, r, \xi, \tau) & \longrightarrow & \mathcal{F}^0(X, x, r, \sigma^* \xi^{-1} \otimes L^{\otimes r}, \tau) \\ (E, \alpha) \longmapsto & & (\sigma^* E^\vee \otimes L, (\alpha^{-1})^t \cdot \alpha_L) \end{array}$$

By the previous discussion, if $\sigma^* \xi^s \otimes L^{\otimes r} \cong \xi$, then $\mathcal{T}_{\sigma, L, s}^0$ is an automorphism of \mathcal{F}^0 . By construction, the map $\overline{\mathcal{T}_{\sigma, L, +}}$ in (5.1.1) is an extension of $\mathcal{T}_{\sigma, L, +}^0$ to the whole moduli space \mathcal{F} . However it will now be shown that a similar extension is not possible for $\mathcal{T}_{\sigma, L, -}^0$ if $r > 2$.

Lemma 5.2.3. *Take $r > 2$, and consider the algebraic automorphism*

$$\begin{array}{ccc} \mathcal{D} : \mathrm{PGL}_r(\mathbb{C}) & \longrightarrow & \mathrm{PGL}_r(\mathbb{C}) \\ [G] \longmapsto & & [(G^{-1})^t]. \end{array}$$

Then there does not exist any algebraic automorphism

$$\overline{\mathcal{D}} : \mathbb{P}(\mathrm{Mat}_r(\mathbb{C})) \rightarrow \mathbb{P}(\mathrm{Mat}_r(\mathbb{C}))$$

extending \mathcal{D} .

Proof. As $\mathrm{PGL}_r(\mathbb{C})$ is dense in $\mathbb{P}(\mathrm{Mat}_r(\mathbb{C}))$ and the latter is irreducible, there exists at most one extension of \mathcal{D} to $\mathbb{P}(\mathrm{Mat}_r(\mathbb{C}))$. Let $\mathcal{U} \subsetneq \mathbb{P}(\mathrm{Mat}_r(\mathbb{C}))$ be the open subset corresponding to matrices with at least an $(r-1) \times (r-1)$ minor with nonzero determinant. Let cof be the morphism that sends each matrix $[G] \in \mathcal{U}$ to its cofactor matrix

$$\mathrm{cof}(G) = \wedge^{r-1}(G).$$

The entries of the cofactor matrix are determinants of minors of G , so they are given by homogeneous polynomials of degree $r-1$ in the entries of G and, therefore, cof induces an algebraic map

$$\mathrm{cof} : \mathcal{U} \longrightarrow \mathbb{P}(\mathrm{Mat}_r(\mathbb{C})).$$

Given an invertible matrix $[G] \in \mathrm{PGL}_r(\mathbb{C})$, we have that

$$(G^{-1})^t = \frac{1}{\det(G)} \mathrm{cof}(G).$$

Therefore, $[(G^{-1})^t] = [\mathrm{cof}(G)]$ for every $[G] \in \mathrm{PGL}_r(\mathbb{C})$ and cof is the unique possible extension of \mathcal{D} to \mathcal{U} . Nevertheless, for $r > 2$ this map is not injective. For example, for every $\lambda \in \mathbb{C}$, let

$$G_\lambda = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ \lambda & 1 & 0 & 0 \\ \hline 0 & 0 & \mathrm{Id}_{r-3} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

Clearly, if $\lambda_1 \neq \lambda_2$ then $[G_{\lambda_1}] \neq [G_{\lambda_2}]$ in $\mathbb{P}(\mathrm{Mat}_r(\mathbb{C}))$. Nevertheless, for every $\lambda \in \mathbb{C}$

$$\mathrm{cof}(G_\lambda) = \left(\begin{array}{cc|c} 0_{r-1} & 0 \\ \hline 0 & 1 \end{array} \right)$$

So, in particular, $[G_\lambda] \in \mathcal{U}$ for every $\lambda \in \mathbb{C}$, which proves that \mathcal{D} cannot be extended to an injective map on \mathcal{U} . \square

Lemma 5.2.4. *If $r > 2$, then the map $\mathcal{T}_{\sigma,L,s}^0 : \mathcal{F}^0 \longrightarrow \mathcal{F}^0$ extends to an automorphism of \mathcal{F} if and only if $s = 1$.*

Proof. Assume that $\mathcal{T}_{\sigma,L,-}^0$ extends to a map $\overline{\mathcal{T}_{\sigma,L,-}}$. For every $E \in \mathcal{M}^s$ and every $(E, \alpha) \in f^{-1}(E) \cap \mathcal{F}^0$, we have

$$f \circ \mathcal{T}_{\sigma,L,-}^0(E, \alpha) = \mathcal{T}_{\sigma,L,-}(E).$$

Since $f^{-1}(E) \cap \mathcal{F}^0$ is dense in $f^{-1}(E)$, for every $(E, \alpha) \in f^{-1}(E)$ we have

$$f \circ \overline{\mathcal{T}_{\sigma,L,-}}(E, \alpha) = \mathcal{T}_{\sigma,L,-}(E).$$

Therefore, it is enough to show that there exists some $E \in \mathcal{M}$ such that the map

$$\mathcal{T}_{\sigma,L,-}^0|_{f^{-1}(E) \cap \mathcal{F}^0} : f^{-1}(E) \cap \mathcal{F}^0 \longrightarrow f^{-1}(\mathcal{T}_{\sigma,L,-}(E))$$

cannot be extended to an isomorphism from $f^{-1}(E)$ to $f^{-1}(\mathcal{T}_{\sigma,L,+}(E))$. Let $E \in \mathcal{M}^s$ be any stable bundle. By [BGM10, Lemma 1.1], the framed bundle (E, α) is τ -stable for every nonzero homomorphism $\alpha : E|_x \rightarrow \mathbb{C}^r$, so

$$f^{-1}(E) = \mathbb{P}(\text{Hom}(E|_x, \mathbb{C}^r)).$$

Then the problem reduces to proving that there does not exist any isomorphism $\mathbb{P}(\text{Hom}(E|_x, \mathbb{C}^r)) \rightarrow \mathbb{P}(\text{Hom}(E|_x^\vee, \mathbb{C}^r))$ extending the map

$$\begin{aligned} \mathbb{P}(\text{Isom}(E|_x, \mathbb{C}^r)) &\longrightarrow \mathbb{P}(\text{Hom}(E|_x^\vee, \mathbb{C}^r)) \\ \alpha &\longmapsto (\alpha^{-1})^t \end{aligned}$$

which, fixing a basis of $E|_x$, is equivalent to proving that there exists no algebraic automorphism $\overline{D} : \mathbb{P}(\text{Mat}_r(\mathbb{C})) \rightarrow \mathbb{P}(\text{Mat}_r(\mathbb{C}))$ extending the transpose of the inverse map

$$\begin{aligned} \text{PGL}_r(\mathbb{C}) &\longrightarrow \text{PGL}_r(\mathbb{C}) \\ \alpha &\longmapsto (\alpha^{-1})^t \end{aligned}$$

Therefore, the result follows from Lemma 5.2.3. \square

The previous results deal with the extension of the maps $\mathcal{T}_{\sigma,L,-}^0$ if $r > 2$. Before proving the main theorem let us address the remaining $r = 2$ case.

Lemma 5.2.5. *Let $r = 2$. Then for every automorphism $\mathcal{T}_{\sigma,L,-} : \mathcal{M} \rightarrow \mathcal{M}$ there exists a line bundle L' on X such that*

$$\mathcal{T}_{\sigma,L,-} = \mathcal{T}_{\sigma,L',+}.$$

Proof. Since $\bigwedge^2 E \cong \xi$ for every $E \in \mathcal{M}$, there is an isomorphism

$$E^\vee \cong E \otimes \xi^{-1}.$$

Consequently, for every σ and L we have

$$\sigma^* E^\vee \otimes L \cong \sigma^*(E \otimes \xi^{-1}) \otimes L \cong \sigma^* E \otimes \sigma^* \xi^{-1} \otimes L.$$

Then taking $L' = \sigma^* \xi^{-1} \otimes L$ yields

$$\mathcal{T}_{\sigma,L,-} = \mathcal{T}_{\sigma,L',+}$$

proving the lemma. \square

5.3 Automorphism group of the moduli space

In this section, we combine the results on the $\text{PGL}_r(\mathbb{C})$ -action on \mathcal{F} proved in [BGM10] with the analysis on the transformations on \mathcal{F} , \mathcal{F}^0 and \mathcal{F}^{ss} given before to prove Theorem 5.0.1 and compute the structure of the automorphism group of \mathcal{F} .

Lemma 5.3.1. *Let $\varphi : \mathcal{F} \rightarrow \mathcal{F}$ be an automorphism. Then there exists an automorphism $\sigma : X \rightarrow X$ with $\sigma(x) = x$, a line bundle L over X and $s \in \{1, -1\}$ satisfying $\sigma^* \xi^s \otimes L^{\otimes r} \cong \xi$, such that the following diagram is commutative*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\varphi} & \mathcal{F} \\ f \downarrow & & \downarrow f \\ \mathcal{M} & \xrightarrow{\mathcal{T}_{\sigma, L, s}} & \mathcal{M} \end{array}$$

Moreover φ preserves both \mathcal{F}^{ss} and \mathcal{F}^0 .

Proof. By [BGM10, Proposition 3.3], there exists an automorphism $\psi : \mathcal{M} \rightarrow \mathcal{M}$ such that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\varphi} & \mathcal{F} \\ f \downarrow & & \downarrow f \\ \mathcal{M} & \xrightarrow{\psi} & \mathcal{M} \end{array}$$

The results by [KP95] and [BGM13] on the structure of the automorphism group of \mathcal{M} imply that there exist an automorphism $\sigma : X \rightarrow X$, a line bundle L over X and $s \in \{1, -1\}$ satisfying $\sigma^* \xi \otimes L^{\otimes r} \cong \xi$ such that $\psi = \mathcal{T}_{\sigma, L, s}$. Moreover, following the argument in [BGM10, Corollary 4.2], as ψ comes from an automorphism of \mathcal{F} , the induced automorphism $\sigma : X \rightarrow X$ must fix the point $x \in X$.

By [BGM10, Proposition 2.5], there exists a unique action of $\mathrm{PGL}_r(\mathbb{C})$ on \mathcal{F} up to a group automorphism of $\mathrm{PGL}_r(\mathbb{C})$. Moreover, by [BGM10, Lemma 3.2] the set of GIT-semistable points for the action of $\mathrm{PGL}_r(\mathbb{C})$ coincides with \mathcal{F}^{ss} for any polarization, so φ restricts to a map $\varphi^{ss} : \mathcal{F}^{ss} \rightarrow \mathcal{F}^{ss}$.

Finally, as $\mathcal{T}_{\sigma, L, s}$ preserves the stable locus $\mathcal{M}^s \subset \mathcal{M}$ for every σ , L and s , it follows that φ preserves $\mathcal{F}^0 = \mathcal{F}^{ss} \cap f^{-1}(\mathcal{M}^s)$. \square

Lemma 5.3.2. *Let $\varphi : \mathcal{F} \rightarrow \mathcal{F}$ be an automorphism. Then there exists $[G] \in \mathrm{PGL}_r(\mathbb{C})$ such that $\varphi_{[G]} \circ \varphi$ is a $\mathrm{PGL}_r(\mathbb{C})$ -equivariant automorphism.*

Proof. Let $\gamma : \mathrm{PGL}_r(\mathbb{C}) \times \mathcal{F} \rightarrow \mathcal{F}$ be the natural action of $\mathrm{PGL}_r(\mathbb{C})$ on \mathcal{F} described before. If φ is an automorphism of \mathcal{F} , it induces another action

$$\gamma' : \mathrm{PGL}_r(\mathbb{C}) \times \mathcal{F} \rightarrow \mathcal{F}$$

given by

$$\gamma'([X], (E, \alpha)) = \varphi(\gamma([X], \varphi^{-1}(E, \alpha))).$$

By [BGM10, Proposition 2.5], there exists a unique action of $\mathrm{PGL}_r(\mathbb{C})$ on \mathcal{F} up to a group automorphism of $\mathrm{PGL}_r(\mathbb{C})$. For $r = 2$, all the automorphisms of $\mathrm{PGL}_2(\mathbb{C})$ are inner and for $r > 2$, the only outer automorphism of $\mathrm{PGL}_r(\mathbb{C})$ is the inverse-transpose, i.e., the map sending $[X] \mapsto [(X^{-1})^t]$. Therefore, there exists a matrix $[G] \in \mathrm{PGL}_r(\mathbb{C})$ such that either

$$\gamma'([X], (E, \alpha)) = \gamma([GXG^{-1}], (E, \alpha)) = \varphi_{[G]}(\gamma([X], \varphi_{[G]}^{-1}(E, \alpha)))$$

or

$$\gamma'([X], (E, \alpha)) = \gamma([G(X^{-1})^t G^{-1}], (E, \alpha)) = \varphi_{[G]}(\gamma([X^{-1}]^t, \varphi_{[G]}^{-1}(E, \alpha)))$$

and it is only necessary to consider the latter when $r > 2$. In the first case, as $\varphi_{[G^{-1}]}$ is an automorphism of \mathcal{F} , it follows that $\varphi_{[G^{-1}]} \circ \varphi$ is a $\mathrm{PGL}_r(\mathbb{C})$ -equivariant automorphism. Let us prove that the second case is impossible if $r > 2$. Let $\mathcal{T}_{\sigma, L, s} : \mathcal{M} \rightarrow \mathcal{M}$ be the automorphism of \mathcal{M} induced by φ . Let E be a stable vector bundle, and let $E' = \mathcal{T}_{\sigma, L, s}(E)$. Then $\varphi_{[G^{-1}]} \circ \varphi$ induces an algebraic isomorphism

$$(\varphi_{[G^{-1}]} \circ \varphi)|_{f^{-1}(E)} : \mathbb{P}(\mathrm{Hom}(E|_x, \mathbb{C}^r)) \rightarrow \mathbb{P}(\mathrm{Hom}(E'|_x, \mathbb{C}^r)).$$

Fix any trivialization $\alpha : E_x \xrightarrow{\sim} \mathbb{C}^r$ of E_x . Let $\alpha' = (\varphi_{[G^{-1}]} \circ \varphi)|_{f^{-1}(E)}(\alpha)$. By Lemma 5.3.1, the composition $\varphi_{[G^{-1}]} \circ \varphi$ preserves \mathcal{F}^0 , so α' is an isomorphism. Using the trivializations α and α' , we get isomorphisms

$$\mathbb{P}(\mathrm{Hom}(E|_x, \mathbb{C}^r)) \xrightarrow{\alpha} \mathbb{P}(\mathrm{Mat}_r(\mathbb{C})) \xrightarrow{\alpha'} \mathbb{P}(\mathrm{Hom}(E'|_x, \mathbb{C}^r));$$

thus $(\varphi_{[G^{-1}]} \circ \varphi)|_{f^{-1}(E)}$ induces an algebraic isomorphism

$$\tilde{\varphi} : \mathbb{P}(\mathrm{Mat}_r(\mathbb{C})) \rightarrow \mathbb{P}(\mathrm{Mat}_r(\mathbb{C})).$$

Moreover, for every $[X] \in \mathrm{PGL}_r(\mathbb{C})$ we have $(\varphi_{[G^{-1}]} \circ \varphi)|_{f^{-1}(E)}(X \circ \alpha) = (X^{-1})^t \alpha'$, so for every $[X] \in \mathrm{PGL}_r(\mathbb{C})$, $\tilde{\varphi}([X]) = [X^{-1}]^t$ and, therefore, $\tilde{\varphi}$ extends the inverse-transpose map to an automorphism of $\mathbb{P}(\mathrm{Mat}_r(\mathbb{C}))$, thus contradicting Lemma 5.2.3. \square

Let \mathbb{P} be the projective bundle over \mathcal{M}^s whose fiber over a stable vector bundle E is $\mathbb{P}(\mathrm{Hom}(E_x, \mathbb{C}^r))$. Even if \mathcal{M}^s does not admit a universal vector bundle, the existence of the bundle \mathbb{P} is guaranteed by [BGM13, Lemma 2.2]. The fiber of its dual bundle \mathbb{P}^\vee over a bundle E is canonically isomorphic to $\mathbb{P}(\mathrm{Hom}(\mathbb{C}^r, E_x))$.

Lemma 5.3.3. *If $r > 2$, then the two projective bundles \mathbb{P} and \mathbb{P}^\vee are not isomorphic.*

Proof. We will break up into several cases because this can be seen from different points of view.

First assume that r and $\deg(\xi)$ are coprime. Then there is a Poincaré vector bundle over $X \times \mathcal{M}^s$. Let

$$W \rightarrow \{x\} \times \mathcal{M}^s = \mathcal{M}^s$$

be the restriction of such a Poincaré bundle to $\{x\} \times \mathcal{M}^s \subset X \times \mathcal{M}^s$. Note that

$$\mathbb{P}^\vee = \mathbb{P}(W^{\oplus r}) \quad \text{and} \quad \mathbb{P} = \mathbb{P}((W^\vee)^{\oplus r}). \quad (5.3.1)$$

Assume that the projective bundles \mathbb{P}^\vee and \mathbb{P} are isomorphic. Consequently, from (5.3.1) it follows that there is a line bundle L_0 on \mathcal{M}^s such that

$$(W^\vee)^{\oplus r} = W^{\oplus r} \otimes L_0. \quad (5.3.2)$$

If A and B are two vector bundles on \mathcal{M}^s such that $A^{\oplus r}$ is isomorphic to $B^{\oplus r}$, then A is isomorphic to B [Ati56, p. 315, Theorem 2]. Therefore, from (5.3.2) it follows that W^\vee is isomorphic to $W \otimes L_0$. Hence the line bundle $\bigwedge^r W^\vee$ is isomorphic to $\bigwedge^r (W \otimes L_0) = L_0^{\otimes r} \otimes \bigwedge^r W$. The Picard group of \mathcal{M}^s is identified with \mathbb{Z} by sending its ample generator to 1 [Ram73]; let $\ell \in \mathbb{Z}$ be the image of $\bigwedge^r W$ by this identification of $\text{Pic}(\mathcal{M}^s)$ with \mathbb{Z} . We have

$$\text{degree}(\xi) \cdot \ell = 1 + ar \quad (5.3.3)$$

for some integer a [Ram73, p. 75, Remark 2.9] (see also [Ram73, p. 75, Definition 2.10]). Since $\bigwedge^r W^\vee = L_0^{\otimes r} \otimes \bigwedge^r W$, we also have

$$-\ell = br + \ell, \quad (5.3.4)$$

where $b \in \mathbb{Z}$ is the image of L_0 . From (5.3.3) and (5.3.4) it follows that

$$2\text{degree}(\xi) \cdot \ell = -\text{degree}(\xi)br = 2 + 2ar.$$

This implies that $r = 2$.

Now assume that r and $\text{degree}(\xi)$ have a common factor. Let

$$\delta = \text{g.c.d.}(r, \text{degree}(\xi)) > 1$$

be the greatest common divisor. The Brauer group $\text{Br}(\mathcal{M}^s)$ of \mathcal{M}^s is the cyclic group $\mathbb{Z}/\delta\mathbb{Z}$, and it is generated by the class of the restriction to $\{x\} \times \mathcal{M}^s$ of the projectivized Poincaré bundle [BBGN07, p. 267, Theorem 1.8]; we will denote this generator of $\text{Br}(\mathcal{M}^s)$ by φ_0 . Now, the class of \mathbb{P}^\vee is φ_0 (tensoring by a vector bundle does not change the Brauer class), and hence the class of \mathbb{P} is $-\varphi_0$. If \mathbb{P}^\vee is isomorphic to \mathbb{P} , then we have $\varphi_0 = -\varphi_0$, hence $\delta = 2$ (as it is the order of φ_0).

We now assume that $\delta = 2$. For a suitable $\mathbb{P}_{\mathbb{C}}^{r-1}$ embedded in \mathcal{M}^s , the restriction of \mathbb{P}^\vee to it is the projectivization of the vector bundle $\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^{r-1}} \oplus \Omega_{\mathbb{P}_{\mathbb{C}}^{r-1}}^1$ [BBPN09, p. 464, Lemma 3.1], [BBPN09, p. 464, (3.4)]; note that any extension of $\Omega_{\mathbb{P}_{\mathbb{C}}^{r-1}}^1$ by $\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^{r-1}}$ splits because $H^1(\mathbb{P}_{\mathbb{C}}^{r-1}, T\mathbb{P}_{\mathbb{C}}^{r-1}) = 0$. Therefore, if \mathbb{P} and \mathbb{P}^\vee are isomorphic, restricting an isomorphism to this embedded $\mathbb{P}_{\mathbb{C}}^{r-1}$ it follows that $\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^{r-1}} \oplus \Omega_{\mathbb{P}_{\mathbb{C}}^{r-1}}^1$ is isomorphic to $(\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^{r-1}} \oplus T\mathbb{P}_{\mathbb{C}}^{r-1}) \otimes L'$ for some line bundle L' on $\mathbb{P}_{\mathbb{C}}^{r-1}$. Since $T\mathbb{P}_{\mathbb{C}}^{r-1}$ is indecomposable, in fact it is stable, from [Ati56, p. 315, Theorem 2] it follows that $T\mathbb{P}_{\mathbb{C}}^{r-1} \otimes L'$ is isomorphic to either $\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^{r-1}}$ or $\Omega_{\mathbb{P}_{\mathbb{C}}^{r-1}}^1$. If $T\mathbb{P}_{\mathbb{C}}^{r-1} \otimes L'$ is isomorphic to $\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^{r-1}}$, then we have $r = 2$. If $T\mathbb{P}_{\mathbb{C}}^{r-1} \otimes L'$ is isomorphic to $\Omega_{\mathbb{P}_{\mathbb{C}}^{r-1}}^1$, we have

$$r + (r - 1) \cdot \text{degree}(L') = -r,$$

so we obtain

$$-(r - 1) \cdot \text{degree}(L') = 2r.$$

Then we conclude that $r - 1$ divides 2, which implies that either $r = 2$ or $r = 3$. However, r is even because $\delta = 2$, so $r = 2$. \square

Lemma 5.3.4. *Let $\varphi : \mathcal{F} \rightarrow \mathcal{F}$ be an automorphism. Then there exist an automorphism $\sigma : X \rightarrow X$ with $\sigma(x) = x$, and a line bundle L over X , such that the induced automorphism on \mathcal{M} is $\mathcal{T}_{\sigma,L,+}$.*

Proof. For $r = 2$, this is a direct consequence of Lemma 5.2.5.

Assume that $r > 2$ and suppose that there exist σ and L such that the induced automorphism on \mathcal{M} is $\mathcal{T}_{\sigma,L,-}$. Let $L' = (\sigma^{-1})^*L$. Then clearly $\mathcal{T}_{\sigma,L,-}^{-1} = \mathcal{T}_{\sigma^{-1},L',-}$. Fix a trivialization $\alpha_L : L_x \xrightarrow{\sim} \mathbb{C}$ and consider the map

$$\begin{aligned} \widetilde{\mathcal{T}_{\sigma^{-1},L',-}} : \text{Tot}(\mathbb{P}) &\xrightarrow{\sim} \text{Tot}(\mathbb{P}^\vee) \\ (E, \alpha) &\longmapsto ((\sigma^{-1})^*E^\vee \otimes L', \alpha^t \otimes \alpha_L^t). \end{aligned}$$

The following diagram is commutative by construction

$$\begin{array}{ccc} \text{Tot}(\mathbb{P}) & \xrightarrow{\widetilde{\mathcal{T}_{\sigma^{-1},L',-}}} & \text{Tot}(\mathbb{P}^\vee) \\ \downarrow & & \downarrow \\ \mathcal{M}^s & \xrightarrow{\mathcal{T}_{\sigma^{-1},L',-}} & \mathcal{M}^s \end{array}$$

Therefore, composing with $\varphi|_{f^{-1}(\mathcal{M}^s)} : \text{Tot}(\mathbb{P}) \xrightarrow{\sim} \text{Tot}(\mathbb{P})$, we obtain an isomorphism $\widetilde{\mathcal{T}_{\sigma^{-1},L',-}} \circ \varphi|_{f^{-1}(\mathcal{M}^s)} : \text{Tot}(\mathbb{P}) \xrightarrow{\sim} \text{Tot}(\mathbb{P}^\vee)$ commuting with the respective projections to \mathcal{M}^s , thus contradicting Lemma 5.3.3. \square

Lemma 5.3.5. *Let $\varphi^0 : \mathcal{F}^0 \rightarrow \mathcal{F}^0$ be a $\text{PGL}_r(\mathbb{C})$ -equivariant automorphism of \mathcal{F}^0 commuting with the forgetful map $f^0 : \mathcal{F}^0 \rightarrow \mathcal{M}^s$. Then φ^0 is the identity map.*

Proof. If φ^0 is $\text{PGL}_r(\mathbb{C})$ -equivariant then it is an automorphism of \mathcal{F}^0 considered as a $\text{PGL}_r(\mathbb{C})$ -principal bundle. Let \mathcal{P} be the universal projective bundle over \mathcal{M}^s , i.e., the unique projective bundle over $X \times \mathcal{M}^s$ whose fiber over each stable vector bundle E is $\mathbb{P}(E)$. Let $\{U_\alpha\}$ be a trivializing cover of \mathcal{M}^s for $\mathcal{P}|_x$, and let $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{PGL}_r(\mathbb{C})$ be the corresponding transition functions. Observe that $\{U_\alpha\}$ is also a trivializing cover for \mathbb{P} and, thus, for the $\text{PGL}_r(\mathbb{C})$ -bundle \mathcal{F}^0 . It is straightforward to check that the transition functions for \mathcal{F}^0 as $\text{PGL}_r(\mathbb{C})$ -bundle are $(g_{\alpha\beta}^{-1})^t$. Therefore, we conclude that \mathcal{F}^0 is the $\text{PGL}_r(\mathbb{C})$ -principal bundle associated to the dual bundle of $\mathcal{P}|_x$, i.e., $\mathcal{P}^\vee|_x$. By [BBPN09], the projective bundle $\mathcal{P}|_x$ is stable and, therefore, its dual $\mathcal{P}^\vee|_x$ must also be stable. Applying the results from [BG08] we know that $\mathcal{P}^\vee|_x$ is simple and, therefore, \mathcal{F}^0 has no nontrivial automorphism, so φ^0 must be the identity map. \square

Theorem 5.3.6. *Let X be a smooth complex projective curve of genus $g > 2$. Assume that $0 < \tau < \tau_0(r)$. Let $\varphi : \mathcal{F} \rightarrow \mathcal{F}$ be an automorphism of the moduli space of τ -semistable framed bundles with fixed determinant ξ . Then there exist*

- an automorphism $\sigma : X \rightarrow X$ with $\sigma(x) = x$,
- a degree zero line bundle $L \in J(X)$ with $\sigma^*\xi \otimes L^{\otimes r} \cong \xi$, and

- a matrix $[G] \in \mathrm{PGL}_r(\mathbb{C})$

such that if we pick any trivialization $\alpha_L : L_x \xrightarrow{\sim} \mathbb{C}$ then for every $(E, \alpha) \in \mathcal{F}$

$$\varphi(E, \alpha) = (\sigma^* E \otimes L, G \circ \alpha \cdot \alpha_L).$$

Proof. By Lemma 5.3.2, composing with $\varphi_{[G]}$ for some $[G] \in \mathrm{PGL}_r(\mathbb{C})$, we may assume without loss of generality that φ is a $\mathrm{PGL}_r(\mathbb{C})$ -equivariant isomorphism. Applying Lemma 5.3.1 and Lemma 5.3.4, there must exist an automorphism $\sigma : X \rightarrow X$ with $\sigma(x) = x$, and a line bundle L over X with $\sigma^* \xi \otimes L^{\otimes r} \cong \xi$, such that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\varphi} & \mathcal{F} \\ f \downarrow & & \downarrow f \\ \mathcal{M} & \xrightarrow{\tau_{\sigma, L, +}} & \mathcal{M} \end{array}$$

Composing with $\overline{\tau_{\sigma, L, +}}^{-1} = \overline{\tau_{\sigma^{-1}, (\sigma^{-1})^* L^{-1}, +}}$, we obtain a map

$$\varphi' = \overline{\tau_{\sigma, L, +}}^{-1} \circ \varphi : \mathcal{F} \rightarrow \mathcal{F}$$

commuting with the projection to \mathcal{M} . The map $\overline{\tau_{\sigma, L, +}}$ is $\mathrm{PGL}_r(\mathbb{C})$ -equivariant by construction, so φ' is a $\mathrm{PGL}_r(\mathbb{C})$ -equivariant automorphism of \mathcal{F} commuting with the projection to \mathcal{M} . By the second part of Lemma 5.3.1, the automorphism φ' preserves \mathcal{F}^0 , so it induces a $\mathrm{PGL}_r(\mathbb{C})$ -bundle map

$$\begin{array}{ccc} \mathcal{F}^0 & \xrightarrow{\varphi^0} & \mathcal{F}^0 \\ & f \searrow \quad \swarrow f & \\ & \mathcal{M}^s & \end{array}$$

Using Lemma 5.3.5 we obtain that φ^0 is the identity map on \mathcal{F}^0 . There exists at most one extension of φ^0 to \mathcal{F} , because \mathcal{F}^0 is dense in \mathcal{F} and the latter is irreducible. Since the identity map of \mathcal{F} is one such extension, it follows that $\varphi' = \mathrm{Id}_{\mathcal{F}}$, so we have $\varphi = \overline{\tau_{\sigma, L, +}}$. \square

Let $J(X)[r]$ denote the r -torsion points in the Jacobian of X , and let $\mathrm{Aut}(X, x)$ be the group of automorphisms of X that fix the point $x \in X$, i.e.,

$$\mathrm{Aut}(X, x) = \{\sigma \in \mathrm{Aut}(X) \mid \sigma(x) = x\}.$$

Corollary 5.3.7. *The automorphism group of \mathcal{F} is*

$$\mathrm{Aut}(\mathcal{F}) \cong \mathrm{PGL}_r(\mathbb{C}) \times \mathcal{T}$$

for a group \mathcal{T} fitting in the short exact sequence

$$1 \rightarrow J(X)[r] \rightarrow \mathcal{T} \rightarrow \mathrm{Aut}(X, x) \rightarrow 1.$$

Proof. We proved that the automorphism group is generated by the maps

- $\varphi_{[G]}$ for each $[G] \in \mathrm{PGL}_r(\mathbb{C})$, and
- $\overline{\mathcal{T}_{\sigma,L,+}}$ for each $\sigma \in \mathrm{Aut}(X, x)$ and each $L \in J(X)$ such that $\sigma^*\xi \otimes L^{\otimes r} \cong \xi$.

First of all, the action of $\mathrm{PGL}_r(\mathbb{C})$ is faithful and commutes with all of the maps $\overline{\mathcal{T}_{\sigma,L,+}}$, so we can split the group $\mathrm{Aut}(\mathcal{F})$ as a product

$$\mathrm{Aut}(\mathcal{F}) \cong \mathrm{PGL}_r(\mathbb{C}) \times \langle \overline{\mathcal{T}_{\sigma,L,+}} \rangle.$$

Observe that, by construction, $\overline{\mathcal{T}_{\sigma,L,+}}$ lies over the automorphism

$$\mathcal{T}_{\sigma,L,+} : \mathcal{M} \longrightarrow \mathcal{M}$$

through the forgetful map $f : \mathcal{F} \longrightarrow \mathcal{M}$. Since the latter is not trivial for any $\sigma \in \mathrm{Aut}(X, x)$ and $L \in \mathrm{Pic}(X)$, apart from $(\sigma, L) = (\mathrm{Id}, \mathcal{O}_X)$, it follows that $\mathcal{T}_{\sigma,L,+} \neq \mathrm{Id}$ for $(\sigma, L) \neq (\mathrm{Id}, \mathcal{O}_X)$. Therefore, in order to obtain the desired result it is enough to prove that the group

$$\mathcal{T} = \langle \{\overline{\mathcal{T}_{\sigma,L,+}}\} \rangle$$

consisting of the maps $\overline{\mathcal{T}_{\sigma,L,+}}$ is an extension of $\mathrm{Aut}(X, x)$ by $J(X)[r]$.

Let $\sigma \in \mathrm{Aut}(X, x)$ be any automorphism. Since $\mathrm{degree}(\sigma^*\xi) = \mathrm{degree}(\xi)$, there is a line bundle $L_\sigma \in J(X)$ such that

$$\sigma^*\xi \otimes L_\sigma^{\otimes r} \cong \xi.$$

Moreover, if $L'_\sigma \in J(X)$ is another line bundle with the same property, then $(L'_\sigma)^{\otimes r} \cong L_\sigma^{\otimes r}$, so L_σ and L'_σ differ by tensoring with an r -torsion element of the Jacobian $J(X)$.

Then, $\langle \overline{\mathcal{T}_{\sigma,L,+}} \rangle$ is generated as a group by the maps

- $\overline{\mathcal{T}_{\sigma,L_\sigma,+}}$ for $\sigma \in \mathrm{Aut}(X, x)$
- $\overline{\mathcal{T}_{\mathrm{Id},L,+}}$ for $L \in J(X)[r]$.

Moreover, for every $\sigma \in \mathrm{Aut}(X, x)$, every $L \in \mathrm{Pic}(X)$ and every $L' \in J(X)[r]$, we have

$$\overline{\mathcal{T}_{\sigma,L,+}} \circ \overline{\mathcal{T}_{\mathrm{Id},L',+}} = \overline{\mathcal{T}_{\mathrm{Id},\sigma^*L',+}} \circ \overline{\mathcal{T}_{\sigma,L,+}}.$$

Since $\sigma^* : J(X)[r] \longrightarrow J(X)[r]$ is an automorphism, it follows that

$$\overline{\mathcal{T}_{\sigma,L,+}} \circ J(X)[r] = J(X)[r] \circ \overline{\mathcal{T}_{\sigma,L,+}}.$$

Therefore, $J(X)[r]$ is a normal subgroup of $\langle \overline{\mathcal{T}_{\sigma,L,+}} \rangle$ and its quotient is precisely $\mathrm{Aut}(X, x)$, so we obtain an exact sequence

$$1 \longrightarrow J(X)[r] \xrightarrow{L \mapsto \overline{\mathcal{T}_{\mathrm{Id},L,+}}} \mathcal{T} \xrightarrow{\overline{\mathcal{T}_{\sigma,L,+}} \mapsto \sigma} \mathrm{Aut}(X, x) \longrightarrow 1$$

This completes the proof. \square

Chapter 6

Conclusions and future work

We started our work on the moduli space of parabolic vector bundles by developing the notion of parabolic Λ -modules as a common theoretical framework for studying moduli spaces of parabolic bundles with additional structures (Parabolic Higgs bundles, parabolic connections, etc.). We built the moduli space of parabolic Λ -modules and, for certain types of sheaves of rings of differential operators Λ , we defined a notion of residue of a Λ -module at a parabolic point generalizing the residue of a logarithmic connection. “Residual Λ -modules” were then defined as Λ -modules with additional restrictions on this residue data mirroring the residual control structures appearing in the Simpson correspondence. A coarse moduli space of “Residual Λ -modules” is then built and this allows us to construct moduli spaces such as

- Strongly parabolic Higgs bundles
- Parabolic connections
- Parabolic λ -connections (parameterized in the parabolic Hodge moduli space)
- In general, parabolic Higgs bundles or connections whose systems of weights and residue eigenvalues is fixed.

This result, combined with an analysis on the regularity of the parabolic Riemann-Hilbert map, allowed us to complete the construction of a parabolic Deligne-Hitchin moduli space described in [AG16, AG18b]. Additionally, the existence of a universal family over these moduli spaces is analyzed, providing a common proof for the fineness of several types of moduli spaces of enhanced parabolic vector bundles under mild conditions on the parabolic weights.

Although the main objective initially stated for this thesis was the computation of the automorphism group of the moduli space of parabolic vector bundles, as we worked on the project we encountered that it was more natural to work on the seemingly more ambitious problem of classifying all possible isomorphisms between moduli spaces of parabolic vector bundles. This change of the point of view - passing from working over a single moduli space $\mathcal{M}(X, r, \alpha, \xi)$ to work intrinsically with maps between two different spaces $\Phi : \mathcal{M}(X, r, \alpha, \xi) \rightarrow \mathcal{M}(X', r', \alpha', \xi')$ - proved to be an effective way of treating the complications in the analysis originated from the existence of a stability parameter which does not exist in the non-parabolic scenario.

Another important distinctive point in our work is the way we treat the discriminant locus and, more generally, the point of view of our analysis on the geometry

of the Hitchin map. The previous approaches to the computation of the automorphism group of the moduli space of vector bundles carried in [HR04], [BGM12] and [BGM13] rely on characterizing the part of the Hitchin discriminant contained in the cotangent of the moduli space $T^*\mathcal{M}(X, r, \xi)$, i.e., the space of Higgs bundles (E, Φ) with stable E which have a singular spectral curve. Then they take the image of this set through the Hitchin map to describe geometrically a subset $\mathcal{D} \subset W$ whose geometry can be used to recover the isomorphism class of the curve X .

Instead, in our analysis we eliminate the intermediate step of finding the Hitchin discriminant upstairs and we focus on determining directly its image \mathcal{D} in W in a geometric way. While in [BGM13] the Hitchin discriminant is recovered as union of complete rational curves in $T^*\mathcal{M}$, in our work we recover \mathcal{D} as the closure of the image of such complete rational curves. This might seem like a minor change, but it decreases significantly the amount of geometrical requirements needed on $T^*\mathcal{M}$, as it is not necessary to recover every single point of the Hitchin discriminant and every single complete rational curve on the cotangent bundle anymore. Instead, we just need to recover *some* complete rational curve landing over points in \mathcal{D} to be able to characterize the latter. Moreover, as we now work on the geometry of \mathcal{D} , we only need to find complete rational curves over a dense open subset of \mathcal{D} , as we can then take the closure in W to recover the whole discriminant \mathcal{D} .

This simplification, in turn, allows us to obtain pretty significant generalizations. For instance, it makes us able to generalize our analysis to the classification of k -birational maps, as well as allowing us to work through stability wall crossings. The general yoga consists on transferring generic behaviors of the geometry of $T^*\mathcal{M}$ to global geometrical properties of the Hitchin space W , thus gaining regularity in the process. The counterpart of this method is that we need additional tools and analysis to get back to the geometry of \mathcal{M} once we obtain our local results. For example, our proof of the main isomorphism classification Theorem 4.6.22 is based on proving that if $\Phi : \mathcal{M} \rightarrow \mathcal{M}'$ is an isomorphism between moduli spaces of parabolic vector bundles and $(E, E_\bullet) \in \mathcal{M}$ is a generic parabolic vector bundle then there exists a basic transformation T such that

$$\Phi(E, E_\bullet) = T(E, E_\bullet)$$

If we stopped the argument here, we would have no information on the global structure of Φ , but rather some pointwise description for generic points. Instead, we are forced to use topological arguments to determine, first, that the structure of Φ when restricted to an open subset of \mathcal{M} does in fact coincide with a single basic transformation T , and then, to consider the possible extensions of that map to the whole moduli space \mathcal{M} .

I would say that this approach to the problem— working locally on the moduli space \mathcal{M} , transferring to global properties of the Hitchin space W and then globalizing the results at the end — has been the key point to the treatment of the stability parameter and the extension of the results to k -birational geometry.

The combination of these strategies has allowed us to classify completely both the isomorphisms and k -birational equivalences between moduli spaces of parabolic vector bundles.

First of all, we have been able to prove a Torelli type theorem for the moduli space of parabolic vector bundles for arbitrary rank, arbitrary determinant and

generic full flag parabolic weights. This result represents a significant improvement with respect to the previous known Torelli type theorem for this moduli space due to Balaji, del Baño and Biswas [BdBnB01], as their result could only be applied under the assumptions of rank 2, degree 1 and small systems of weights. Moreover, we unlock other Torelli type results which were restricted to these latter hypothesis because they used the result in [BdBnB01]. For instance, proving this more general version of the Torelli theorem allows us extend the Torelli type theorems proved in [AG18b] for the following moduli spaces to arbitrary rank, arbitrary determinant and generic full flag weights

- Moduli space of parabolic Higgs bundles
- Parabolic Hodge moduli space (moduli space of parabolic λ -connections constructed in Chapter 3)
- Parabolic Deligne–Hitchin moduli space

Then, using this Theorem as a basis, we have classified all possible isomorphisms $\Phi : \mathcal{M}(X, r, \alpha, \xi) \longrightarrow \mathcal{M}(X', r', \alpha', \xi')$, proving that they are all essentially obtained as combinations of four types of transformations

1. Tensoring with a line bundle
2. Taking the pullback with respect to an isomorphism which preserves the set of parabolic points
3. Taking the parabolic dual
4. Performing a Hecke transformation at the parabolic points

This allowed us to refine the Torelli theorem previously obtained, effectively classifying the possible isomorphism classes of moduli spaces of parabolic vector bundles in terms of their parameters and stating a reciprocal for the Torelli theorem (i.e., determining precisely when two moduli spaces of parabolic vector bundles are isomorphic or not). As a consequence of these results, we have also been able to solve the Torelli problem for the fixed degree situation, generalizing the Torelli type theorem proved by Biswas, Gómez and Logares [BGL16].

Then we have been able to extend these results to the classification k -birational equivalences between the moduli spaces. After studying the geometry of the moduli spaces and the structure of the wall crossings, we determined that the basic transformations of quasi-parabolic vector bundles are not always isomorphisms, but instead they are naturally k -birational equivalences. Thus, the k -birational equivalence class of a moduli space of parabolic vector bundles seems to contain more useful geometrical information than the isomorphism class itself (specially regarding the dependence on the geometry of the curve). In fact, we have proven that the 3-birational classes of moduli spaces of parabolic vector bundles are in bijective correspondence with the isomorphism classes of the corresponding marked curves together with a choice of a rank. This contrasts to the classification result on the isomorphism classes of the moduli spaces, as we proved that there exist several non-isomorphic moduli spaces over the same curve corresponding to different choices of the stability parameters and the degree of the determinant.

We have complemented this analysis with a full description of the stability chambers for high genus curves. We computed a numerical invariant providing a full classification of the numerical chambers for any genus and then we used a combination of Brill Noether theory and the description of the stratification of the moduli space of stable parabolic vector bundles in terms of the Segre invariant proved by Bhosle and Biswas [BB05] to characterize some geometrical chambers. As a consequence, we are given computable explicit descriptions of the automorphism group of the moduli space of parabolic vector bundles in two scenarios: concentrated weights or high genus curves.

Summing up, we have solved completely the strong and refined versions of the Torelli problem for the moduli space of parabolic vector bundles with fixed determinant and the refined Torelli theorem for the moduli space of parabolic vector bundles with fixed degree. Moreover, we have been able to prove that analogous theorems (strong and refined Torelli) can still be stated if we substitute the isomorphism between the moduli spaces by a k -birational equivalence.

Finally, the experience and strategies developed during the exploration and posterior solution of the main problem have allowed us to transfer some of the techniques to other similar problems of computation of automorphisms of moduli spaces. In particular, we have been able to export some of the general ideas underlying the analysis of the isomorphisms between moduli spaces of parabolic vector bundles to the computation of the automorphism group of the moduli space of framed bundles, proving that they are generated by the following transformations

- Tensoring by a line bundle
- Taking the pullback with respect to an isomorphism preserving the marked point
- Changing the framing by changing the basis of \mathbb{C}^r through multiplication by a fixed matrix in $\mathrm{PGL}_r(\mathbb{C})$

To conclude this thesis, I would like to make some comments regarding future projects and lines of work which emerged or were inspired from some of the analysis previously presented.

6.1 Parabolic Λ -modules in higher dimension

First of all I shall address the extension of the framework of parabolic Λ -modules to higher dimension. The reader could have noticed that most of the technical lemmas and the general strategy for the construction of the moduli space of parabolic Λ -modules could potentially work for higher dimensional varieties, but the main results are only stated on curves. I would like to mention that, indeed, the whole construction can be rewritten for a higher dimensional variety. In fact, for the most part, I originally developed it for higher dimensional varieties and then the statements and proofs were simplified for the case of curves. Clearly, this involves a slightly different computation for the equivalence between the (now Gieseker) stability of the parabolic Λ -module and the GIT-stability of its representative in the parameter space, as the GIT stability computed here has been further simplified form

the general framework using the fact that we were working on a curve. Nevertheless, essentially all the stated results hold true for higher dimensional varieties under mild conditions.

The main reason why these more general results do not to appear in this thesis is not of technical nature, but rather an issue of deducing the “correct” definition of parabolic Λ -modules for a higher dimensional variety – in the sense of choosing the most useful and natural one in cases where the sheaf of differential operators Λ has special interactions with the parabolic divisor. It seems that there could be two possible natural definitions for the parabolic structure.

1. Component-wise, i.e., asking for a filtration by subbundles on each component of the parabolic divisor, i.e., for each $D_i \subset D$

$$E|_{D_i} = E_{D_i,1} \supsetneq E_{D_i,2} \supsetneq \cdots E_{D_i,l_i} \supsetneq 0$$

together with a sequence of real numbers $0 \leq \alpha_1(D_i) \leq \cdots \leq \alpha_{l_i}(D_i) < 1$.

2. Asking for a single filtration by subsheaves on the whole divisor

$$E \supsetneq E_1 \supsetneq \cdots \supsetneq E_l \supsetneq E(-D)$$

together with a sequence of real numbers $0 \leq \alpha_1 \leq \cdots \leq \alpha_l < 1$.

In the case of curves, both definitions are essentially equivalent, as different components of D are simply different points and, therefore, they are disjoint. Conversely, if X has higher dimension and the components D_i of D cross, both definitions are not equivalent at all. Moreover, it is not clear whether one should impose further conditions between the filtrations (or, more precisely, the divisors) and Λ in order to obtain a natural object. We have seen part of this issue when treating the residual structures in dimension one through Section 3.5, but the problem gets more convoluted as we increase the dimension, as other more complex structures might appear in Λ . For example, if we take a foliated manifold X , we may consider Λ to be the sheaf of differentials along the foliation. For this choice of Λ , the Λ -modules would be vector bundles with a “horizontal” (in the foliation sense) connection. If we took an arbitrary divisor in X , what should the parabolic structure of one such Λ -module be? For example, what happens if D is a leaf of the foliation? Would it be interesting to check for connections that are logarithmic “along” a leaf? Or does it make more sense to restrict our attention to connections that are singular over a locus transverse to the leaves? Do parabolic connections in any of these senses have an interesting counterpart in representation theory?

In brief, I would like to remark that, if no other conditions on the filtration were taken into consideration, the construction described in Chapter 3 can be generalized to build a moduli space of Λ -modules carrying a compatible filtration of the form (1) or (2) over varieties of any dimension. Nevertheless, it is not clear that these objects are as useful as their counterparts in dimension 1 if no other compatibility conditions are imposed (such as higher dimensional residual conditions), so I believe that further study on the “singular” operators on higher dimension would be needed before defining and constructing an analogue of the moduli space in higher dimension. This way, we ensure that the resulting scheme has an actual geometrical meaning and utility.

6.2 Automorphisms of the moduli stack of parabolic vector bundles

Regarding the computation of automorphism groups, a question arising naturally is what is the difference between the automorphism groups of a moduli space and the moduli stack of parabolic vector bundles. As it is clear that basic transformations are well defined on arbitrary families of quasi-parabolic vector bundles, then they clearly induce automorphisms on the moduli stack. On the other hand, when we study the moduli stack instead of the moduli space the stability parameter effectively disappears from the analysis, and so does most of our concerns regarding the stability of the quasi-parabolic vector bundles obtained through a basic transformation.

Thanks to the change of point of view regarding our use of the geometry of the Hitchin map in this work, most of the techniques developed for the analysis of the moduli space can be transferred to characterize automorphisms of the moduli stack. As we mentioned earlier, we have reduced most of the crucial arguments in the proof of the classification Theorems (4.3.6, 4.6.22, 4.7.10, etc.) to either questions regarding the geometry of the Hitchin space alone or the cotangent space at generic points. For example, contrary to other similar works such as [BGM12] or [BGM13], our proof of the fact that any isomorphism $\Phi : \mathcal{M} \rightarrow \mathcal{M}'$ induces a linear map $f : W \rightarrow W'$ between the corresponding Hitchin spaces relies only on the geometry of the image of the discriminant locus inside the Hitchin space $\mathcal{D} \subset W$ and, more particularly, on the geometry of the locus of non-reduced spectral curves $\mathcal{N} \subset \mathcal{D} \subset W$. For this reason, we believe that the techniques developed in this work are also suitable for analyzing the automorphisms of the moduli stack and we have started working on this line of research, obtaining some promising preliminary results.

6.3 Automorphisms of moduli spaces of principal G -bundles

On the other hand, it is natural to wonder if similar results could be found for the automorphism group of moduli spaces of principal bundles. Biswas, Gómez and Muñoz proved it to be the case for symplectic bundles [BGM12], and Sancho found similar results for F_4 and E_6 -bundles [Sán18]. In both cases, the scheme of the proof is based on the same idea developed by Biswas, Gómez and Muñoz for the moduli space of vector bundles [BGM13]: prove that if $\Phi : \mathcal{M} \rightarrow \mathcal{M}$ is an automorphism of the moduli space of stable principal G -bundles, then for a generic bundle E we have an isomorphism of Lie algebra bundles

$$E(\mathfrak{g}) \cong \Phi(E)(\mathfrak{g})$$

where \mathfrak{g} is the Lie algebra of G . Then, the classification is based on the computation of the possible reductions of $\text{Aut}(\mathfrak{g})$ -principal structures to G -structures which, at the end, mainly relies on computing the outer automorphisms of $\text{Aut}(\mathfrak{g})$. Nevertheless, while this proof sketch seems completely clear, it is not straightforward to extend each particular detail of the proof to an arbitrary group G , as the structures of the moduli space and the Hitchin map change drastically as soon as we step away

from $\mathrm{SL}_r(\mathbb{C})$ (or similar groups like $\mathrm{Sp}(r)$). Nevertheless, we believe that some of the simplifications and the new strategies developed in this work to cope with the additional parabolic structures can be also used to surpass these difficulties and address effectively the classification theorem for the isomorphisms between moduli spaces of principal G -bundles for other groups.

Chapter 7

Conclusiones y futuros proyectos

Comenzamos nuestro trabajo sobre el espacio de moduli de fibrados parabólicos desarrollando la noción de Λ -módulo parabólico como un marco teórico común para el estudio de espacios de moduli de fibrados parabólicos con estructuras adicionales (fibrados de Higgs parabólicos, conexiones parabólicas, etc.). Construimos el espacio de moduli de Λ -módulos parabólicos y, para ciertos tipos de haces de anillos de operadores diferenciales Λ , definimos una noción de residuo de un Λ -módulo en un punto parabólico que generaliza el residuo de una conexión logarítmica. Entonces, definimos los “ Λ -módulos residuales” como Λ -módulos con restricciones adicionales sobre su residuo que se asemejan al tipo de control sobre el residuo que aparece en la correspondencia de Simpson. Construimos un espacio de moduli grueso de “ Λ -módulos residuales” lo que, a su vez, nos permite construir otros espacios de moduli tales como:

- Fibrados de Higgs (fuertemente) parabólicos
- Conexiones parabólicas
- λ -conexiones parabólicas (parametrizadas por el espacio de moduli de Hodge parabólico)
- En general, fibrados de Higgs parabólicos o conexiones parabólicas cuyos sistemas de pesos y autovalores de su residuo hayan sido fijados

Este resultado, combinado con un análisis sobre la regularidad de la correspondencia de Riemann–Hilbert parabólica, nos permiten completar la construcción del espacio de Deligne–Hitchin parabólico descrita en [AG16, AG18b]. Además, analizamos la existencia de una familia universal sobre estos espacios de moduli, proporcionando una demostración común para la existencia de familias universales sobre diversos tipos de espacios de moduli de fibrados parabólicos con estructuras adicionales bajo condiciones moderadas en los pesos parabólicos.

Aunque el principal objetivo establecido inicialmente para esta tesis era el cálculo del grupo de automorfismos del espacio de moduli de fibrados parabólicos, a medida que trabajábamos en el proyecto encontramos que resultaba más natural trabajar con el problema (aparentemente más complejo) de clasificar todos los posi-

bles isomorfismos entre espacios de moduli de fibrados parabólicos. Este cambio en la orientación del problema (pasar de trabajar sobre un único espacio de moduli $\mathcal{M}(X, r, \alpha, \xi)$ a trabajar intrínsecamente sobre dos espacios distintos $\Phi : \mathcal{M}(X, r, \alpha, \xi) \rightarrow \mathcal{M}(X', r', \alpha', \xi')$) resultó ser una manera efectiva de tratar con las complicaciones en el análisis del problema inicial originadas por la existencia de un parámetro de estabilidad que no estaba presente en el escenario no parabólico.

Otro punto diferenciador importante en nuestro trabajo es la manera en que tratamos el discriminante y, más generalmente, la manera de afrontar nuestro análisis sobre la geometría del morfismo de Hitchin. Los enfoques anteriores sobre el cálculo del grupo de automorfismos llevados a cabo en [HR04], [BGM12] y [BGM13] se basan en caracterizar la parte del discriminante de Hitchin contenida en el cotangente del moduli $T^*\mathcal{M}(X, r, \xi)$, es decir, el espacio de fibrados de Higgs (E, Φ) con E estable cuya curva espectral es singular. Entonces, tomar la imagen de este conjunto a través del morfismo de Hitchin permite describir geométricamente un subconjunto $\mathcal{D} \subset W$ cuya geometría puede usarse para recuperar la clase de isomorfismo de la curva X .

Por el contrario, en nuestro análisis eliminamos el paso intermedio de identificar el discriminante de Hitchin incluido en el cotangente y, en su lugar, nos centramos en determinar directamente de forma geométrica su imagen \mathcal{D} en W . Mientras que en [BGM13] el discriminante de Hitchin se recupera como la unión de curvas racionales completas en $T^*\mathcal{M}$, en nuestro trabajo recuperamos \mathcal{D} como el cierre de la imagen de tales curvas racionales completas. Esto puede parecer un cambio pequeño, pero reduce significativamente los requisitos geométricos sobre $T^*\mathcal{M}$ necesarios, al no ser ya necesario recuperar cada punto del discriminante de Hitchin y todas y cada una de las curvas racionales completas en el cotangente. En su lugar, únicamente necesitamos recuperar *algunas* curvas racionales completas que caigan sobre puntos de \mathcal{D} para poder caracterizar este último conjunto. Además, como ahora trabajamos con la geometría de \mathcal{D} , únicamente tenemos que encontrar curvas racionales sobre un subconjunto denso de \mathcal{D} , ya que podemos tomar el correspondiente cierre en W para recuperar el discriminante completo \mathcal{D} .

En última instancia, esta simplificación nos permite obtener algunas generalizaciones bastante significativas. Por ejemplo, nos permite generalizar nuestro análisis a la clasificación de aplicaciones k -birracionales, así como operar a través de las barreras de estabilidad. El mantra general consiste en transferir comportamientos y propiedades que suceden genéricamente en la geometría de $T^*\mathcal{M}$ a propiedades geométricas globales del espacio de Hitchin W , ganando por lo tanto regularidad en el proceso. La contrapartida de este método es que necesitamos herramientas y análisis adicionales para volver a la geometría de \mathcal{M} una vez hemos obtenido los resultados locales. Por ejemplo, nuestra prueba del teorema principal de clasificación de isomorfismos 4.6.22 está basada en probar que si $\Phi : \mathcal{M} \rightarrow \mathcal{M}'$ es un isomorfismo entre espacios de moduli de fibrados parabólicos y $(E, E_\bullet) \in \mathcal{M}$ es un fibrado parabólico genérico entonces existe una transformación básica T tal que

$$\Phi(E, E_\bullet) = T(E, E_\bullet)$$

Sin embargo, si parásemos el argumento en este punto, no conseguiríamos información sobre la estructura global de Φ , si no simplemente una descripción punto a punto para puntos genéricos. En su lugar, estamos forzados a utilizar argumentos

topológicos adicionales para determinar, por un lado, que la estructura de Φ coincide efectivamente con una única transformación básica cuando se restringe a un abierto de \mathcal{M} y, además, a considerar las posibles extensiones de esta aplicación al espacio de moduli total \mathcal{M} .

En mi opinión, esta aproximación al problema (trabajar localmente con el espacio de moduli \mathcal{M} para después transferir los resultados a propiedades globales del espacio de Hitchin W y globalizar de nuevo los resultados al final) ha sido la clave para un tratamiento adecuado del parámetro de estabilidad y para la extensión de los resultados a geometría k -birracional. La combinación de estas estrategias nos ha permitido clasificar completamente tanto los isomorfismos como las equivalencias k -birracionales entre espacios de moduli de fibrados vectoriales parabólicos.

En primer lugar, hemos sido capaces de probar un teorema tipo Torelli para el espacio de moduli de fibrados parabólicos para rango y determinante arbitrarios y pesos completos (full flag) genéricos. Este resultado representa una mejora significativa respecto a los teoremas tipo Torelli conocidos hasta este punto debidos a Balaji, del Baño y Biswas [BdBnB01], ya que sus resultado únicamente podía ser aplicado bajo las hipótesis de rango 2, grado 1 y sistema de pesos pequeño. Además, desbloqueamos otros resultados tipo Torelli que permanecían restringidos a estas últimas hipótesis debido a que utilizaban el resultado en [BdBnB01]. Por ejemplo, demostrar esta versión más general del teorema de Torelli nos permite extender los teoremas tipo Torelli probados en [AG18b] para los siguientes espacios de móduli a rango arbitrario, cualquier determinante y pesos genéricos completos (full flag):

- Espacio de moduli de fibrados parabólicos de Higgs
- Espacio de moduli de Hodge parabólico (espacio de moduli de λ -conexiones parabólicas construido en el Capítulo 3
- Espacio de Deligne–Hitchin parabólico

Entonces, utilizando este Teorema como base, clasificamos todos los posibles isomorfismos $\Phi : \mathcal{M}(X, r, \alpha, \xi) \rightarrow \mathcal{M}(X', r', \alpha', \xi')$, demostrando que, esencialmente, todos pueden obtenerse como combinaciones de las siguientes cuatro tipos de transformaciones:

1. Tensorizar con un fibrado de línea
2. Tomar el pullback con respecto a un isomorfismo que respete el conjunto de puntos parabólicos
3. Tomar el dual parabólico
4. Realizar una transformada de Hecke en puntos parabólicos

Esto nos permite refinar el teorema de Toerlli previamente obtenido, clasificando de manera efectiva las posibles clases de isomorfismo de espacios de moduli de fibrados vectoriales parabólicos en términos de sus parámetros y estableciendo un recíproco para el teorema de Torelli (es decir, determinando de manera precisa cuándo dos espacios de moduli de fibrados vectoriales parabólicos son isomorfos o no). Como consecuencia de estos resultados, también hemos sido capaces de resolver el problema

de Torelli para el escenario con grado fijo, generalizando el teorema tipo Torelli probado por Biswas, Gómez y Logares [BGL16].

A partir de aquí, hemos podido extender estos resultados a la clasificación de equivalencias k -birracional entre espacios de moduli. Tras estudiar la geometría de los espacios de moduli y la estructura de los cruces de barreras (wall crossings), determinamos que las transformaciones básicas de fibrados vectoriales cuasiparabólicos no son siempre isomorfismos, sino que en su lugar representan de forma natural equivalencias k -birracional. Por tanto, las clases de equivalencia k -birracional del espacio de moduli de fibrados vectoriales parabólicos contienen, en apariencia, más información geométrica útil que la propia clase de isomorfismo (especialmente en lo que respecta a la dependencia con la geometría de la curva). De hecho, hemos probado que las clases de equivalencia 3-birracional de los espacios de moduli de fibrados vectoriales parabólicos están en correspondencia biyectiva con los pares formados por las clases de isomorfismo de las correspondientes curvas marcadas y un rango. Esto contrasta con el teorema de clasificación de las clases de isomorfismo de los espacios de moduli, ya que probamos que existen varios espacios de moduli no isomorfos sobre la misma curva correspondientes a distintas elecciones de los parámetros de estabilidad y del grado del determinante.

Complementamos este análisis con una descripción completa de las cámaras de estabilidad para curvas de género alto. Calculamos un invariante numérico que proporciona una clasificación plena de las cámaras numéricas para cualquier género y entonces usamos una combinación de teoría de Brill Noether y la descripción de la estratificación del espacio de moduli de fibrados vectoriales parabólicos estables en términos del invariante de Segre proporcionada por Bhosle y Biswas [BB05] para caracterizar algunas cámaras geométricas. Como consecuencia, se obtienen descripciones explícitas y computables del grupo de automorfismos del espacio de moduli de fibrados vectoriales parabólicos en dos escenarios: pesos concentrados o curvas de género alto.

Resumiendo, hemos resuelto completamente las versiones fuerte y refinada del problema de Torelli para el espacio de moduli de fibrados vectoriales parabólicos con determinante fijo y el teorema de Torelli refinado para el espacio de moduli de fibrados vectoriales parabólicos con determinante fijo. Además, hemos podido probar que se obtienen teoremas análogos (Torelli fuerte y refinado) si sustituimos el isomorfismo entre los espacios de moduli por una equivalencia k -birracional.

Finalmente, la experiencia adquirida y estrategias desarrolladas durante la exploración y posterior solución del problema principal nos ha permitido transferir algunas de estas técnicas a otros problemas similares de cálculo de automorfismos de espacios de moduli. En particular, hemos podido exportar algunas de las ideas generales subyacentes al análisis de los isomorfismos entre espacios de moduli de fibrados vectoriales parabólicos al cálculo del grupo de automorfismos del espacio de moduli de fibrados marcados, demostrando que están generados por las siguientes transformaciones:

1. Tensorizar por un fibrado de línea
2. Tomar el pullback respecto a un isomorfismo que preserva el punto marcado
3. Cambiar el marcado mediante un cambio de base de \mathbb{C}^r a través de la multi-

plicación con una matriz fija en $\mathrm{PGL}_r(\mathbb{C})$

Concluiré esta tesis realizando algunos comentarios adicionales respecto a futuros proyectos y líneas de investigación que han emergido de los análisis previamente presentados o se inspiraron en ellos.

7.1 Λ -módulos parabólicos en dimensión superior

En primer lugar, me gustaría abordar la extensión del marco de trabajo de Λ -módulos parabólicos a dimensión superior. El lector puede haber notado que la mayor parte de los lemas técnicos y la estrategia general de la construcción del espacio de moduli de Λ -módulos parabólicos podría potencialmente funcionar también para variedades de dimensión superior, aunque los resultados principales únicamente se enuncien para curvas. Quisiera mencionar que, en realidad, toda la construcción puede reescribirse para una variedad de dimensión superior. De hecho, originalmente desarrollé la mayor parte de dicha construcción para variedades de dimensión superior y más adelante los enunciados y pruebas fueron simplificados para el caso de curvas. Claramente esto involucra un cálculo ligeramente distinto para la equivalencia entre la estabilidad (ahora Gieseker) del Λ -módulo parabólico y la estabilidad GIT de su representante en el espacio de parámetros, ya que la estabilidad GIT calculada aquí ha sido tremendamente simplificada desde el marco general utilizando el hecho de que estamos trabajando sobre una curva. Sin embargo, esencialmente todos los resultados enunciados se mantienen para variedades de dimensión superior incorporando ciertas hipótesis moderadas.

La principal razón por la que estos resultados más generales no aparecen en la tesis no es de carácter técnico, sino un tema de deducir la definición “correcta” de Λ -módulo parabólico para una variedad de dimensión superior, en el sentido de escoger la más útil y natural en los casos en los que el haz de operadores diferenciales Λ tiene interacciones especiales con el divisor parabólico. Aparentemente podría haber dos posibles definiciones naturales para la estructura parabólica:

1. Componente a componente, es decir, requerir una filtración por subfibrados sobre cada componente del divisor. En otras palabras, para cada $D_i \subset D$

$$E|_{D_i} = E_{D_i,1} \supsetneq E_{D_i,2} \supsetneq \cdots E_{D_i,l_i} \supsetneq 0$$

junto con una sucesión de números reales $0 \leq \alpha_1(D_i) \leq \cdots \leq \alpha_{l_i}(D_i) < 1$.

2. Requerir una única filtración por subhaces sobre el divisor completo

$$E \supsetneq E_1 \supsetneq \cdots \supsetneq E_l \supsetneq E(-D)$$

junto con una sucesión de números reales $0 \leq \alpha_1 \leq \cdots \leq \alpha_l < 1$.

En el caso de curvas, ambas definiciones son esencialmente equivalentes, ya que componentes distintas de D son simplemente puntos distintos y, por tanto, son disjuntos. Por el contrario, si X tiene dimensión superior y las componentes D_i de D se cruzan, ambas definiciones no son para nada equivalentes. Además, no está claro si se debería imponer alguna condición adicional sobre las filtraciones (o,

más precisamente, los divisores) y Λ para obtener un objeto más natural. Hemos vislumbrado parte de este problema cuando tratábamos las estructuras residuales a lo largo de la Sección 3.5, pero el problema se vuelve más intrincado a medida que aumentamos la dimensión, ya que otras estructuras más complejas pueden aparecer en Λ . Por ejemplo, si tomamos una variedad foliada X , podemos considerar Λ como el haz de operadores diferenciales a lo largo de la foliación. Para esta elección de Λ , los Λ -módulos serían fibrados vectoriales con una conexión “horizontal” (en el sentido de la foliación). Si tomamos un divisor arbitrario en X , ¿cuál debería ser la estructura parabólica para uno de esos Λ -módulos? Por ejemplo, ¿qué sucede si D es una hoja de la foliación? ¿Sería acaso interesante tratar las conexiones que son logarítmicas “a lo largo” de la hoja? ¿O tiene más sentido restringir nuestra atención a conexiones que son singulares sobre un conjunto transversal a las hojas? ¿Las conexiones parabólicas en alguno de estos sentidos tiene alguna contrapartida interesante en teoría de la representación?

Resumiendo, me gustaría remarcar que, si no se tuvieran en consideración otras condiciones sobre la filtración, la construcción descrita en el Capítulo 3 podría generalizarse para construir un espacio de Λ -módulos dotados de una filtración compatible de al forma (1) o (2) para variedades de cualquier dimensión. Sin embargo, no está claro que estos objetos sean tan útiles como sus homólogos de dimensión 1 si no se imponen condiciones de compatibilidad adicionales (como, por ejemplo, condiciones residuales de dimensión superior), así que creo que es necesario un estudio más profundo sobre los operadores “singulares” en dimensión superior antes de definir y construir un análogo del espacio de moduli para dimensión alta. De esta manera, garantizamos que el esquema resultante tenga un verdadero significado y utilidad geométrica real.

7.2 Automorfismos del stack de moduli de fibrados vectoriales parabólicos

En cuanto al cálculo de grupos de automorfismos, una cuestión que surge de forma natural es cuál es la diferencia entre los grupos de automorfismos de un espacio de moduli de fibrados vectoriales parabólicos y el stack de moduli completo. Está claro que las transformaciones básicas están bien definidas en familias arbitrarias de fibrados quasi-parabólicos y, por tanto, claramente inducen automorfismos del stack de moduli. Por otro lado, cuando estudiamos el stack de moduli en lugar del espacio de moduli, el parámetro de estabilidad desaparece a nivel práctico del análisis, así como la mayoría de nuestras preocupaciones acerca de la estabilidad de los fibrados quasi-parabólicos obtenidos mediante las transformaciones básicas.

Gracias al cambio en el punto de vista respecto a nuestro uso de la geometría del morfismo de Hitchin llevado a cabo en este trabajo, muchas de las técnicas desarrolladas para el análisis del espacio de moduli pueden transferirse para caracterizar automorfismos del stack de moduli. Como mencionamos anteriormente, hemos reducido la mayoría de los argumentos cruciales de la prueba de los Teoremas de Clasificación (4.3.6, 4.6.22, 4.7.10, etc.) o bien a cuestiones concernientes únicamente a la geometría del espacio de Hitchin, o bien a la del cotangente al moduli en puntos genéricos. Por ejemplo, el hecho de que cualquier isomorfismo

$\Phi : \mathcal{M} \rightarrow \mathcal{M}'$ induce una aplicación lineal $f : W \rightarrow W'$ entre los correspondientes espacios de Hitchin se basa únicamente en la geometría de la imagen del discriminante dentro del espacio de Hitchin $\mathcal{D} \subset W$ y, más concretamente, en la geometría del lugar geométrico de curvas espectrales no reducidas $\mathcal{N} \subset \mathcal{D} \subset W$. Por este motivo, creemos que las técnicas desarrolladas en este trabajo también son válidas para analizar los automorfismos del stack de moduli y ya hemos comenzado a trabajar en esta línea de investigación, obteniendo resultados preliminares prometedores.

7.3 Automorfismos de espacios de moduli de G -fibrados principales

Por otro lado, es natural preguntarse si se podrían obtener resultados similares para el grupo de automorfismos de espacios de moduli de fibrados principales. Biswas, Gómez y Muñoz probaron que éste es el caso para fibrados simplécticos [BGM12], y Sancho encontró resultados similares para fibrados principales con grupo F_4 y E_6 [Sán18]. En ambos casos, el esquema de la demostración está basado en la misma idea desarrollada por Biswas, Gómez y Muñoz para el espacio de moduli de fibrados vectoriales [BGM13]: demostrar que si $\Phi : \mathcal{M} \rightarrow \mathcal{M}$ es un automorfismo del espacio de moduli de G -fibrados principales estables entonces para un fibrado genérico E tenemos un isomorfismo de fibrados de álgebras de Lie

$$E(\mathfrak{g}) \cong \Phi(E)(\mathfrak{g})$$

donde \mathfrak{g} es el álgebra de Lie de G . Entonces, la clasificación se basa en el cálculo de las posibles reducciones de $\text{Aut}(\mathfrak{g})$ -estructuras principales a G -estructuras, lo que, en última instancia, depende de calcular el grupo de automorfismos externos de $\text{Aut}(\mathfrak{g})$. Sin embargo, aunque el esquema de esta demostración parece bastante claro, no resulta inmediato extender cada detalle de la prueba a un grupo arbitrario G , ya que algunas de las estructuras del espacio de moduli y el morfismo de Hitchin cambian drásticamente en cuanto nos alejamos de $\text{SL}_r(\mathbb{C})$ (o grupos similares como $\text{Sp}(r)$). Sin embargo, creemos que algunas de las simplificaciones y nuevas estrategias desarrolladas en este trabajo para lidiar con las estructuras parabólicas adicionales pueden utilizarse también para sobrepasar estas dificultades y afrontar de manera efectiva un teorema de clasificación de isomorfismos entre espacios de moduli de G -fibrados para otros grupos.

Appendices

Appendix A

Sheaves of bimodules

Through chapter 3 (and specially in the part regarding residual structures on parabolic Λ -modules), it is necessary to work with sheaves of bi-modules in an intrinsically non-commutative setup. While the algebraic theory of bi-modules over non-commutative rings is completely classic, there are some basic results about sheaves of bi-modules over shaves of non-commutative rings needed for the development of the theory for which I could not find any references in the literature. The objective of this appendix is to provide proofs for the needed lemmata. I am almost sure that the results exposed here are well known, but lacking a better reference, I will include them for completeness.

A.1 Bimodules and tensor product

Proposition A.1.1. *Let R, S, T, U be rings and let*

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$$

be a short exact sequence of (S, T) -modules and let M be a (R, S) -module. Then

$$M \otimes_S N_1 \longrightarrow M \otimes_S N_2 \longrightarrow M \otimes_S N_3 \longrightarrow 0$$

is an exact sequence of (R, T) -modules, i.e., $M \otimes_S -$ is right exact. Similarly, if M' is a (T, U) -module, then

$$N_1 \otimes_T M' \longrightarrow N_2 \otimes_T M' \longrightarrow N_3 \otimes_T M' \longrightarrow 0$$

is an exact sequence of (S, U) -modules, i.e., $- \otimes_T M'$ is right exact.

Proof. By [AF73, Proposition 19.13] we obtain that $M \otimes_S -$ and $- \otimes_T M'$ are exact functors to the space of left R -modules and right U -modules respectively. As the maps are right T -linear and left S -linear respectively, we obtain exact sequences of (R, T) and (S, U) -modules respectively. \square

Lemma A.1.2. *Given rings R and S and modules ${}_S U_R$ and ${}_S N$, there is a natural isomorphism*

$$\mathrm{Hom}_S(U \otimes_R -, N) \xrightarrow{\sim} \mathrm{Hom}_R(-, \mathrm{Hom}_S(U, N))$$

which is moreover natural in N , i.e., the functors $U \otimes_R -$ and $\mathrm{Hom}_S(U, -)$ are adjoint.

Proof. See [AF73, Lemma 20.6]. \square

Let \mathcal{I} be a directed system. Let $\{R_i\}_{\mathcal{I}}$, $\{S_i\}_{\mathcal{I}}$ and $\{T_i\}_{\mathcal{I}}$ be directed systems of rings. For each $i \leq j$ let

$$\begin{aligned}\varphi_{i \rightarrow j}^R &: R_i \rightarrow R_j \\ \varphi_{i \rightarrow j}^S &: S_i \rightarrow S_j \\ \varphi_{i \rightarrow j}^T &: T_i \rightarrow T_j\end{aligned}$$

be the restriction morphisms. For each $i \in \mathcal{I}$ let M_i be a (R_i, S_i) -module and let N_i be a (S_i, T_i) -module such that if $i \leq j$ there are morphisms of (R_i, S_i) -modules $\varphi_{i \rightarrow j}^M : M_i \rightarrow M_j$ and (S_i, T_i) -modules $\varphi_{i \rightarrow j}^N : N_i \rightarrow N_j$ respectively. Let

$$\begin{aligned}R &= \varinjlim R_i \\ S &= \varinjlim S_i \\ T &= \varinjlim T_i \\ M &= \varinjlim M_i \\ N &= \varinjlim N_i\end{aligned}$$

Then the following lemmata hold.

Lemma A.1.3. *M is a (R, S) -module and it is the direct limit of M_i in the category of bimodules.*

Proof. For all $i < j$, the bimodule M_j acquires a structure of (R_i, S_i) -module through the morphisms $R_i \rightarrow R_j$ and $S_i \rightarrow S_j$. Let us prove that M is a (R, S) -module for every $i \in \mathcal{I}$. Let $i \in \mathcal{I}$. Let $r \in R_i$ and $s \in S_i$. For every $[m] \in M$ there exist a representative $m_j \in M_j$ for some $j > i$. Therefore, r and s act on m_j and we can define

$$\begin{aligned}r \cdot [m] &= [r \cdot m_j] \\ [m] \cdot s &= [m_j \cdot s]\end{aligned}$$

To check that it is a well defined action, suppose that $m_k \in M_k$ is another representative for $k > i$. If there are both representatives for $[m]$, then there exist a $l \in \mathcal{I}$, with $l \geq j$ and $l \geq k$ such that

$$\varphi_{j \rightarrow l}^M(m_j) = \varphi_{k \rightarrow l}^M(m_k) = m_l$$

Then as the morphisms $\varphi_{j \rightarrow l}^M : M_j \rightarrow M_l$ are morphisms of (R_i, S_i) -modules we obtain

$$\varphi_{j \rightarrow l}^M(r \cdot m_j) = r \cdot \varphi_{j \rightarrow l}^M(m_j) = r \cdot m_l = \varphi_{k \rightarrow l}^M(r \cdot m_k)$$

Therefore $[r \cdot m_j] = [r \cdot m_l] = [r \cdot m_k]$. The proof for the right action of S_i is analogous. And the (R_i, S_i) -module properties (associative, bilinear, etc.) follow from the properties of (R_i, S_i) -module of M_j .

Moreover, observe that if $\varphi_{i \rightarrow j}^R(r_i) = r_j$, then if $m_k \in M_k$ with $k \geq j$, yields

$$r_i \cdot [m_k] = [r_i \cdot m_k] = [\varphi_{i \rightarrow k}^R(r_i) \cdot m_k] = [\varphi_{j \rightarrow k}^R(r_j) \cdot m_k] = [r_j \cdot m_k] = r_j \cdot [m_k]$$

Therefore, R acts on M as

$$[r_i] \cdot [m] = r_i \cdot [m]$$

Similarly, S acts on M as

$$[m] \cdot [s_i] = [m] \cdot s_i$$

and the (R, S) -module structure is derived from the (R_i, S_i) -structure. Now, let us prove that it corresponds to the limit of $R_i M_i S_i$ in the category of bimodules. Let ${}_A X_B$ be a bimodule such that there are morphisms

$$\alpha_i : R_i \longrightarrow A$$

$$\beta_i : S_i \longrightarrow B$$

$$\varphi_i : M_i \longrightarrow X$$

such that m_i is a morphism of (R_i, S_i) -modules. Then as $R = \varinjlim R_i$ and $S = \varinjlim S_i$ we have morphisms of rings

$$\alpha : R \rightarrow A$$

$$\beta : S \rightarrow B$$

and a morphism of groups

$$\varphi : M \longrightarrow X$$

such that α_i, β_i and φ_i factor through the corresponding morphisms. To prove the Lemma it is enough to show that φ is a morphism of (R, S) -modules. Let $[m] \in M$ and $[r] \in R$, then for some representatives $r_j \in R_j$ and $m_j \in M_j$ yields

$$\varphi([r] \cdot [m]) = \varphi([r_j \cdot m_j]) = \varphi_j(r_j \cdot m_j) = \alpha_j(r_j) \cdot \varphi_j(m_j) = \alpha([r]) \cdot \varphi([m])$$

Therefore, φ is R -linear and we can prove analogously that it is S -linear. \square

Proposition A.1.4.

$$\varinjlim M_i \otimes_{S_i} N_i = M \otimes_S N$$

as (R, T) -modules.

Proof. By the previous lemma, the left R and right T module structures on the limit are the desired ones. Therefore, it is enough to prove the equality as groups. Now we can adapt a construction by G. Sassatelli.

For every $i \in \mathcal{I}$ let $\varphi_i^S : S_i \rightarrow S$. Then we have a morphism

$$f_i : M_i \otimes_{S_i} N_i \longrightarrow M \otimes_{S_i} N \twoheadrightarrow M \otimes_S N$$

The map clearly commutes with restrictions $\varphi_{i \rightarrow j}^M \otimes \varphi_{i \rightarrow j}^N$, so it descends to the direct limit

$$f : \varinjlim M_i \otimes_{S_i} N_i \longrightarrow M \otimes_S N$$

On the other hand, for every $i \in \mathcal{I}$, we have an S_i -balanced morphism

$$g_i : M_i \times N_i \longrightarrow M_i \otimes_{S_i} N_i \longrightarrow \varinjlim M_i \otimes_{S_i} N_i$$

As the direct limit commutes with the Cartesian product, taking direct limit at the left hand side yields a morphism

$$\tilde{g} : \varinjlim (M_i \times N_i) = (\varinjlim M_i) \times (\varinjlim N_i) = M \times N \longrightarrow \varinjlim M_i \otimes_{S_i} N_i$$

Let us prove that \tilde{g} is S -balanced. Let $[m] \in M$, $[n] \in N$, $[s] \in S$. For some $i \in \mathcal{I}$ and any representatives $m_i \in M$, $n_i \in N$, $s_i \in S_i$ we have

$$\begin{aligned} \tilde{g}([m] \cdot [s], [n]) &= \tilde{g}([m_i \cdot s_i], [n_i]) = g_i(m_i \cdot s_i, n_i) = g_i(m_i, s_i \cdot n_i) \\ &= \tilde{g}([m_i], [s_i \cdot n_i]) = \tilde{g}([m], [s] \cdot [n]) \end{aligned}$$

as g_i is S_i -balanced. As \tilde{g} is an S -balanced morphism, by the universal property of the tensor product, it factors through a morphism

$$g : M \otimes_S N \longrightarrow \varinjlim M_i \otimes_{S_i} N_i$$

By construction f and g are both homomorphisms, so in order to prove that they are isomorphisms of groups it is enough to prove that they are inverse of each other. Let $[m] \otimes [n] \in M \otimes_S N$. There is some $i \in \mathcal{I}$ and representatives $m_i \in M_i$ and $n_i \in N_i$ such that $[m] \otimes [n] = [m_i] \otimes [n_i]$. By definition of g

$$g([m_i] \otimes [n_i]) = g_i(m_i, n_i) = \pi_i(m_i \otimes n_i)$$

where $\pi_i : M_i \otimes_{S_i} N_i \longrightarrow \varinjlim M_i \otimes_{S_i} N_i$. On the other hand, by construction of f yields

$$f(\pi_i(m_i \otimes n_i)) = f_i(m_i \otimes n_i) = [m_i] \otimes [n_i]$$

So $f \circ g = \text{Id}_{M \otimes_S N}$. Now let $\pi_i(m_i \otimes n_i) \in \varinjlim M_i \otimes_{S_i} N_i$ for some $i \in \mathcal{I}$. By definition of f

$$f(\pi_i(m_i \otimes n_i)) = f_i(m_i \otimes n_i) = [m_i] \otimes [n_i]$$

Then

$$g([m_i] \otimes [n_i]) = \tilde{g}([m_i], [n_i]) = g_i(m_i, n_i) = \pi_i(m_i \otimes n_i)$$

As this holds for every $i \in \mathcal{I}$ then $g \circ f = \text{Id}_{\varinjlim M_i \otimes_{S_i} N_i}$. □

Given a morphism of rings $f : R \rightarrow S$ and a right R -module (respectively left R -module) M , we denote

$$M_S = M \otimes_R S \quad (\text{resp. } {}_S M = S \otimes_R M)$$

Let N be an R -module, i.e., a (R, R) -module such that the left and right R -module structures coincide. Then N_S (respectively ${}_S N$) can be provided a left (respectively right) S -module structure taking the same S -module structure at both sides. Through the canonical isomorphism of groups $N_S \cong {}_S N$, both (S, S) -module structures coincide. Let us denote by ${}_S N_S$ the corresponding S -bimodule.

Lemma A.1.5. *Let $f : R \rightarrow S$ be a morphism of commutative rings and let T be a ring. Let M be a (T, R) -module and let N be an R -module, i.e., a (R, R) -module such that the left and right actions of R are the same. Then there is a canonical isomorphism*

$$(M \otimes_R N)_S = M_S \otimes_S {}_S N_S$$

Proof. By associativity of tensor product we have

$$(M \otimes_R N) \otimes_R S = M \otimes_R (N \otimes_R S) = M \otimes_R N_S$$

Now we can use that both R -module structures of N are equivalent to prove that N_S and ${}_S N_S$ are equal as (R, S) -modules. As they share the same right S -module structure by construction, it is enough to prove that the left R -module structure is the same. Let $r \in R$, $s \in S$ and $n \in N$. Then

$$r \cdot {}_S N_S (n \otimes_R s) = (rn) \otimes_R s = (nr) \otimes_R s = n \otimes_R (f(r)s) = n \otimes_R (s \cdot r) = r \cdot {}_S N_S (n \otimes_R s)$$

Therefore, substituting and applying again associativity of tensor product we obtain

$$M \otimes_R N_S = M \otimes_R {}_S N_S = M \otimes_R (S \otimes_S {}_S N_S) = (M \otimes_R S) \otimes_S {}_S N_S = M_S \otimes_S {}_S N_S$$

□

A.2 Sheaves of bimodules

Definition A.2.1. Let \mathcal{A} and \mathcal{B} be sheaves of rings over X . A sheaf of $(\mathcal{A}, \mathcal{B})$ -modules is a sheaf of groups \mathcal{M} such that for every open set $U \subseteq X$, the group $\mathcal{M}(U)$ is a $(\mathcal{A}(U), \mathcal{B}(U))$ -bimodule and for every $V \subseteq U$ the restriction morphism $\mathcal{M}(U) \rightarrow \mathcal{M}(V)$ is a morphism of $(\mathcal{A}(U), \mathcal{B}(U))$ -bimodules, where $\mathcal{M}(V)$ is considered as a $(\mathcal{A}(U), \mathcal{B}(U))$ -bimodule through the morphisms

$$\begin{aligned} \mathcal{A}(U) &\rightarrow \mathcal{A}(V) \\ \mathcal{B}(U) &\rightarrow \mathcal{B}(V) \end{aligned}$$

A morphism of $(\mathcal{A}, \mathcal{B})$ -modules is a morphism of sheaves of groups $\varphi : \mathcal{M} \rightarrow \mathcal{N}$ such that $\varphi(U) : \mathcal{M}(U) \rightarrow \mathcal{N}(U)$ is a morphism of $(\mathcal{A}(U), \mathcal{B}(U))$ -bimodules. If \mathcal{M} is a $(\mathcal{A}, \mathcal{B})$ -bimodule, we will write ${}_A \mathcal{M}_B$.

Given sheaves of $(\mathcal{A}, \mathcal{B})$ -modules, \mathcal{M} and \mathcal{N} , we define the sheaf $\underline{\text{Hom}}^l(\mathcal{M}, \mathcal{N})$ as

$$\underline{\text{Hom}}^l(\mathcal{M}, \mathcal{N})(U) = \text{Hom}_{\mathcal{A}(U)}(\mathcal{M}(U), \mathcal{N}(U))$$

i.e., it is the sheaf of local morphisms of left \mathcal{A} -modules from \mathcal{M} to \mathcal{N} . Similarly, we define the sheaf $\underline{\text{Hom}}^r(\mathcal{M}, \mathcal{N})$ as

$$\underline{\text{Hom}}^r(\mathcal{M}, \mathcal{N})(U) = \text{Hom}_{\mathcal{B}(U)}(\mathcal{M}(U), \mathcal{N}(U))$$

i.e., it is the sheaf of local morphisms of right \mathcal{B} -modules.

Let \mathcal{A} , \mathcal{B} , \mathcal{C} and \mathcal{D} be sheaves of rings over X .

Proposition A.2.2. Let $f : X \rightarrow Y$ be a morphism. Let ${}_A \mathcal{M}_B$. Then $f^{-1}(\mathcal{M})$ is a $(f^{-1}(\mathcal{A}), f^{-1}(\mathcal{B}))$ -module.

Proof. Let $V \subseteq Y$. Then for every open subset $U \subseteq X$ with $f(U) \subseteq V$, $\mathcal{M}(U)$ is a $(\mathcal{A}(U), \mathcal{B}(U))$ -module. By Lemma A.1.3, the presheaf associated to the inverse image $f_{\text{pre}}^{-1}(\mathcal{M})(V) = \varinjlim \mathcal{M}(U)$ is a $(\varinjlim \mathcal{A}(U), \varinjlim \mathcal{B}(U)) = (f_{\text{pre}}^{-1}(\mathcal{A})(V), f_{\text{pre}}^{-1}(\mathcal{B})(V))$ -module.

Taking direct limits, for every $y \in Y$, the stalk

$$f_{\text{pre}}^{-1}(\mathcal{M})_y = \varinjlim_{p \in V} f_{\text{pre}}^{-1}(\mathcal{M})(V)$$

is a bimodule for

$$(\varinjlim_{p \in V} f_{\text{pre}}^{-1}(\mathcal{A})(V), \varinjlim_{p \in V} f_{\text{pre}}^{-1}(\mathcal{B})(V)) = (f_{\text{pre}}^{-1}(\mathcal{A})_y, f_{\text{pre}}^{-1}(\mathcal{B})_y)$$

Therefore, taking the associated sheaf, $f^{-1}(\mathcal{M})$ is a $(f^{-1}(\mathcal{A}), f^{-1}(\mathcal{B}))$ -module. \square

Corollary A.2.3. *Let $f : X \rightarrow Y$ be a morphism and let Λ be a sheaf of rings over X . Then $f^{-1}(\Lambda)$ is a sheaf of rings over Y .*

Proof. If Λ is a sheaf of rings then Λ is a (Λ, Λ) -module. By the previous proposition $f^{-1}(\Lambda)$ is a $(f^{-1}(\Lambda), f^{-1}(\Lambda))$ -module. As the left and right Λ -module structures correspond to the same underlying product on Λ , the $(f^{-1}(\Lambda), f^{-1}(\Lambda))$ -module structure induces an associative product on $f^{-1}(\Lambda)$. \square

Lemma A.2.4. *Let $\mathcal{A}\mathcal{M}_{\mathcal{B}}$ and ${}_{\mathcal{B}}\mathcal{N}_{\mathcal{C}}$. Then for every $x \in X$,*

$$(\mathcal{M} \otimes_{\mathcal{B}} \mathcal{N})_x = \mathcal{M}_x \otimes_{\mathcal{B}_x} \mathcal{N}_x$$

and it is a $(\mathcal{A}_x, \mathcal{C}_x)$ -module.

Proof. By Lemma A.1.4 we have

$$\begin{aligned} (\mathcal{M} \otimes_{\mathcal{B}} \mathcal{N})_x &= \varinjlim_{x \in U} (\mathcal{M} \otimes_{\mathcal{B}} \mathcal{N})(U) = \varinjlim_{x \in U} \mathcal{M}(U) \otimes_{\mathcal{B}(U)} \mathcal{N}(U) \\ &= \left(\varinjlim_{x \in U} \mathcal{M}(U) \right) \otimes_{\left(\varinjlim_{x \in U} \mathcal{B}(U) \right)} \left(\varinjlim_{x \in U} \mathcal{N}(U) \right) = \mathcal{M}_x \otimes_{\mathcal{B}_x} \mathcal{N}_x \end{aligned}$$

and it is a $(\mathcal{A}_x, \mathcal{C}_x)$ -module. \square

Proposition A.2.5. *Let*

$$0 \longrightarrow \mathcal{N}_1 \longrightarrow \mathcal{N}_2 \longrightarrow \mathcal{N}_3 \longrightarrow 0$$

be a short exact sequence of $(\mathcal{B}, \mathcal{C})$ -modules and let \mathcal{M} be a $(\mathcal{A}, \mathcal{B})$ -module. Then

$$\mathcal{M} \otimes_{\mathcal{B}} \mathcal{N}_1 \longrightarrow \mathcal{M} \otimes_{\mathcal{B}} \mathcal{N}_2 \longrightarrow \mathcal{M} \otimes_{\mathcal{B}} \mathcal{N}_3 \longrightarrow 0$$

is an exact sequence of $(\mathcal{A}, \mathcal{C})$ -modules, i.e., $\mathcal{M} \otimes_{\mathcal{B}} -$ is right exact. Similarly, if \mathcal{M}' is a $(\mathcal{C}, \mathcal{D})$ -module, then

$$\mathcal{N}_1 \otimes_{\mathcal{C}} \mathcal{M}' \longrightarrow \mathcal{N}_2 \otimes_{\mathcal{C}} \mathcal{M}' \longrightarrow \mathcal{N}_3 \otimes_{\mathcal{C}} \mathcal{M}' \longrightarrow 0$$

is an exact sequence of $(\mathcal{B}, \mathcal{D})$ -modules, i.e., $- \otimes_{\mathcal{C}} \mathcal{M}'$ is right exact.

Proof. Taking stalks at each $x \in X$ this is a consequence of the previous lemma and Proposition A.1.1. \square

Proposition A.2.6. *Let Λ be a sheaf of $(\mathcal{O}_X, \mathcal{O}_X)$ -algebras, i.e., an associative left- \mathcal{O}_X and right- \mathcal{O}_X algebra. Let E be a \mathcal{O}_X -module and let $i : D \hookrightarrow X$ be a simple effective divisor on X . If E has a left action of Λ that preserves $E(-D)$ then $i_R^*(\Lambda) := i^{-1}(\Lambda) \otimes_{i^{-1}(\mathcal{O}_X)} \mathcal{O}_X$ acts on $E|_D$.*

Proof. Consider the short exact sequence of \mathcal{O}_X -modules

$$0 \longrightarrow E(-D) \longrightarrow E \xrightarrow{\text{ev}} i_*E|_D \longrightarrow 0$$

Tensoring by the $(\mathcal{O}_X, \mathcal{O}_X)$ -module Λ we get an exact sequence

$$\Lambda \otimes_{\mathcal{O}_X} E(-D) \longrightarrow \Lambda \otimes_{\mathcal{O}_X} E \longrightarrow \Lambda \otimes_{\mathcal{O}_X} i_*E|_D \longrightarrow 0$$

The left Λ -module structure on E is given by a morphism of $(\mathcal{O}_X, \mathcal{O}_X)$ -modules

$$\varphi : \Lambda \otimes_{\mathcal{O}_X} E \longrightarrow E$$

First, let us prove that if this morphism preserves $E(-D)$ then it induces an action on $i_*E|_D$. We have the following commutative diagram

$$\begin{array}{ccccccc} \Lambda \otimes_{\mathcal{O}_X} E(-D) & \longrightarrow & \Lambda \otimes_{\mathcal{O}_X} E & \longrightarrow & \Lambda \otimes_{\mathcal{O}_X} i_*E|_D & \longrightarrow & 0 \\ \downarrow & & \downarrow \varphi & & & & \\ 0 & \longrightarrow & E(-D) & \longrightarrow & E & \xrightarrow{\text{ev}} & i_*E|_D \longrightarrow 0 \end{array}$$

Given $\lambda \otimes v \in \Lambda \otimes_{\mathcal{O}_X} i_*E|_D$, let $\bar{v} \in \text{ev}^{-1}(v)$. Define

$$\varphi|_D : \Lambda \otimes_{\mathcal{O}_X} i_*E|_D \rightarrow i_*E|_D$$

as $\varphi|_D(\lambda \otimes v) = \text{ev}(\varphi(\lambda \otimes v))$. To check that it is well defined, observe that if we take $\bar{v}' \in \text{ev}^{-1}(v)$ then $\bar{v}' - \bar{v} \in E(-D)$. Therefore

$$\varphi(\lambda \otimes \bar{v}') - \varphi(\lambda \otimes \bar{v}) = \varphi(\lambda \otimes (\bar{v}' - \bar{v})) \in E(-D)$$

So

$$0 = \text{ev}(\varphi(\lambda \otimes \bar{v}') - \varphi(\lambda \otimes \bar{v})) = (\text{ev} \circ \varphi)(\lambda \otimes \bar{v}') - (\text{ev} \circ \varphi)(\lambda \otimes \bar{v})$$

Taking the preimage yields

$$i^{-1}\varphi|_D : i^{-1}(\Lambda) \otimes_{i^{-1}(\mathcal{O}_X)} i^{-1}i_*E|_D \longrightarrow i^{-1}i_*E|_D$$

tensoring on the right by \mathcal{O}_D and applying Lemma A.1.5 yields

$$i^*\varphi|_D : i_R^*(\Lambda) \otimes_{\mathcal{O}_D} E|_D \longrightarrow E|_D$$

□

Appendix B

The category of parabolic vector bundles

This appendix gathers some of the main definitions and results regarding parabolic vector bundles that are used through the rest of the thesis. It does not intend to be an exhaustive description of the theory of parabolic sheaves, but rather it aims to serve as a reminder of some of the main constructions in the category of filtered vector bundles that are used in this work and to provide a comparison between the main different existing formalisms for parabolic vector bundles in the literature.

In this thesis we introduce and work with several equivalent frameworks for parabolic bundles, opting for the most appropriate formalism depending on our needs. This appendix serves as a common meeting point for these definitions, as we provide explicit equivalences between the formalisms and deepen in the presented constructions, analyzing them from the point of view of different descriptions of the parabolic category.

We start by recalling the main definitions of parabolic vector bundle and filtered vector bundle used in the literature. We will focus on describing equivalences between the different descriptions, analyzing whether the described parabolic objects truly codify the same information or whether, on the contrary, there exists any significant difference between them which leads to effectively different categories of parabolic objects. As the rest of the thesis is mainly centered around parabolic vector bundles on curves, we will concentrate in a comparison for dimension one, but we will also identify differences between formalisms that appear only when we consider parabolic structures over divisors on higher dimensional varieties.

The definition of parabolic homomorphism and the notion of strongly parabolic map will be introduced, completing the structure of the category of parabolic vector bundles. Then we will comment some internal operations in this category, namely the internal parabolic Hom (as a parabolic sheaf), the direct sum of parabolic bundles, the parabolic version of the tensor product, the shifting operation and the dual of a parabolic vector bundle. The objective of this part is twofold. On one hand, we provide some additional insight to several constructions that appear through the thesis (specially in Chapter 4) and, on the other hand, we exemplify how these constructions can be further described and analyzed by re-interpreting them in the framework of the appropriate formalism of parabolic vector bundles.

Finally, we will give some remarks about parabolic Serre duality. The main

references used for this section are [Yok95], [Bis03] and [IS08].

B.1 Filtered vector bundles

In the literature we can find different formalisms for describing parabolic structures on vector bundles and sheaves on varieties with a marked divisor. Most of the alternative definitions give rise to equivalent categories in dimension one but depending on the type of properties in which we are interested, it may be more simple or practical to work with one of them in particular. On the other hand, on higher dimensional varieties there are certain additional subtleties that ought to be taken into account, as we may need to impose additional conditions in order to obtain equivalent categories (see, for example [IS08, Section 2]). In this section we will explain several of the main descriptions of categories of parabolic bundles...

As we are mainly interested on parabolic vector bundles on curves, for simplicity we shall restrict ourselves to dimension 1. Let X be a smooth curve and let $D = \{x_1, \dots, x_n\}$ be a set of marked points on D . Let $U = X \setminus D$ and let $i : U \hookrightarrow X$ be the inclusion.

Parabolic vector bundles on curves were first described by Mehta and Seshadri [MS80]. They define a quasi-parabolic parabolic vector bundles as follows.

Definition B.1.1 (Quasi-parabolic vector bundle Mehta-Seshadri). *A quasi-parabolic vector bundle on (X, D) is a vector bundle E on X endowed with a filtration on the fiber $E|_x$ over each parabolic point $x \in D$*

$$E|_x = E_{x,1} \supsetneq E_{x,2} \supsetneq \dots \supsetneq E_{x,l_x} \supsetneq E_{x,l_x+1} = 0$$

A parabolic vector bundle is then described as a quasi-parabolic structure together with some real weights associated to the filtrations.

Definition B.1.2 (Parabolic vector bundle Mehta-Seshadri). *A parabolic structure on E is a quasi-parabolic structure together with a set of real weights*

$$0 \leq \alpha_1(x) < \alpha_2(x) < \dots < \alpha_{l_x}(x) < 1$$

We say that $\alpha_i(x)$ is the weight associated to $E_{x,i}$. We will call $\alpha = \{(\alpha_1(x), \dots, \alpha_{l_x}(x))\}_{x \in D}$ the system of weights of the parabolic structure. We say that a quasi-parabolic vector bundle or a system of weights is full flag if $l_x = r$ for all parabolic points $x \in D$. We write $(E, E_\bullet) = (E, \{E_{x,i}\})$ to denote a parabolic vector bundle (or, with a slight abuse of notation, the underlying quasi-parabolic bundle). We say that a parabolic vector bundle (E, E_\bullet) is of type $\bar{n} = (n_i(x))$ if

$$n_i(x) = \dim(E_{x,i}) - \dim(E_{x,i+1})$$

and call $n_i(x)$ the multiplicity of $\alpha_i(x)$.

The filtrations at the parabolic points can be codified alternatively using subsheaves $F \subset E$ of the vector bundle E such that the quotient E/F is supported on the parabolic divisor D . There have been different approaches to this matter, being specially fruitful the descriptions by Maruyama-Yokogawa [MY92], Simpson [Sim90] and Mochizuki [Moc06] (also used and explored by Iyer and Simpson [IS08]).

Starting with the notion of filtered vector bundle by Simpson [Sim90], we can codify the parabolic structure as a collection of decreasing left continuous filtrations of sheaves on extensions of $E|_U$ to $U \cup \{x\}$ for each $x \in D$. More precisely, for each $x \in D$, let $\widetilde{E}_\alpha^x \subset E$ be a subsheaf on $U \cup \{x\}$ indexed by a real parameter α such that

1. For every $\alpha \geq \beta$, $\widetilde{E}_\alpha^x \subseteq \widetilde{E}_\beta^x$
2. For every $\alpha \in \mathbb{R}$, there exists $\varepsilon > 0$ such that $\widetilde{E}_{\alpha-\varepsilon}^x = \widetilde{E}_\alpha^x$
3. For every α , $\widetilde{E}_{\alpha+1}^x = \widetilde{E}_\alpha^x(-x)$
4. $\widetilde{E}_0^x = E|_{U \cup \{x\}}$

Definition B.1.3 (Parabolic vector bundle Simpson). *We call a vector bundle E on X together with a set of sheaves $\{\widetilde{E}_\alpha^x\}_{\alpha \in \mathbb{R}}$ over $U \cup \{x\}$ for each $x \in D$ satisfying (1)-(4) a parabolic vector bundle on (X, D) .*

Observe that, by construction, if $i_x : U \rightarrow U \cup \{x\}$, then

$$(i_x)_* i_x^* E = \bigcup_{\alpha \in \mathbb{R}} \widetilde{E}_\alpha^x$$

Moreover, for every $x, y \in D$ and every $\alpha, \beta \in \mathbb{R}$, \widetilde{E}_α^x and \widetilde{E}_β^y agree on U , so we can glue them together to obtain extensions to the whole curve X . In particular, if we extend \widetilde{E}_α^x using \widetilde{E}_0^y for all $x \in D$ for $\alpha \geq 0$, we obtain a set of subsheaves

$$E_\alpha^x = \widetilde{E}_\alpha^x \cup \bigcup_{y \in D \setminus \{x\}} \widetilde{E}_0^y \hookrightarrow E$$

satisfying analogous conditions

- 1' For every $\alpha \geq \beta$, $E_\alpha^x \subseteq E_\beta^x$
- 2' For every $\alpha > 0$, there exists $\varepsilon > 0$ such that $E_{\alpha-\varepsilon}^x = E_\alpha^x$
- 3' For every α , $E_{\alpha+1}^x = E_\alpha^x(-x)$
- 4' $E_0^x = E$

Clearly both structures give us the same information. We can pass from one to the other simply by gluing the filtrations or restricting them respectively and we obtain the following alternative description

Definition B.1.4 (Parabolic vector bundle Simpson (alternative)). *We call a vector bundle E together with a set of filtrations $\{E_\alpha^x\}_{\alpha \in \mathbb{R}}$ over X for each $x \in D$ satisfying (1')-(4') a parabolic vector bundle on (X, D) .*

For convenience, in this thesis, apart from using the Mehta-Seshadri Definition B.1.2, we will work with this alternative definition B.1.4 instead of the one described originally by Simpson in [Sim90] (Definition B.1.3). In any case, all definitions give rise to equivalent categories of parabolic vector bundles in the following way. Given

a left continuous filtration E_α^x , for every $x \in D$ consider the set of values $\alpha \in [0, 1]$ such that the filtration E_α^x “jumps”, i.e., such that for every $\varepsilon > 0$, $E_\alpha^x \neq E_{\alpha+\varepsilon}^x$. As the filtration is left-continuous, the rank of E is finite and we have the periodicity condition $E_1^x = E_0^x(-x)$, we know that there are finitely many values of α between 0 and 1 where this can happen. For each $x \in D$ let $\alpha_i(x)$ be the i -th positive value where the filtration jumps. Then we can construct a parabolic structure $\{E_{x,i}\}$ at the fiber $E|_x$ in the sense of Definition B.1.2 as the one having parabolic weights $\{\alpha_i(x)\}$ and such that

$$E|_x/E_{x,i} \otimes \mathcal{O}_x = E/E_{\alpha_i(x)}^x$$

Reciprocally, let $\{E_{x,i}\}$ be a filtration of the fiber $E|_x$, endowed with weights $\alpha_i(x)$. Then we can define the subsheaves $E_{\alpha_i(x)}^x \subseteq E$ as the ones fitting in the short exact sequence

$$0 \longrightarrow E_{\alpha_i(x)}^x \longrightarrow E \longrightarrow E/E_{x,i} \otimes \mathcal{O}_x \longrightarrow 0$$

As $\{E_{x,i}\}$ form a filtration of $E|_x$, clearly the sheaves $E_{\alpha_i(x)}^x$ are a filtration of E

$$E = E_{\alpha_1(x)}^x \supsetneq E_{\alpha_2(x)}^x \supsetneq \dots \supsetneq E_{\alpha_l(x)}^x \supsetneq E(-x)$$

Now we can extend this filtration to a filtration E_α^x indexed by $\alpha \in \mathbb{R}$ as the only one satisfying (1')-(4'). Take $E_\alpha^x = E$ for $\alpha_{l_x}(x) - 1 \leq \alpha \leq \alpha_1(x)$ and $E_\alpha^x = E_{\alpha_i(x)}^x$ for $\alpha_{i-1}(x) < \alpha \leq \alpha_i(x)$ and define E_α^x for $\alpha > \alpha_{l_x}(x)$ using the periodicity condition

$$E_{\alpha+1}^x = E_\alpha^x(-x)$$

By construction, the set of resulting filtrations E_α^x for each $x \in D$ is a parabolic structure at the point x in the sense of definition B.1.4.

Through the thesis we have been alternating between the previous definitions, using the one that was more suitable for each situation. The definition of the parabolic structure of a parabolic Λ -module from Chapter 3 was based on a combination of Definition B.1.2 and Definition B.1.4. The families of quasi-parabolic structures were more appropriately described through a Mehta-Seshadri-like definition, while the control of the interaction between the action of the sheaf of rings of differential operators Λ and the parabolic structure is better described through the usage of the Simpson's formalism (Definition B.1.4).

With regards to the analysis of the isomorphisms between moduli spaces of parabolic vector bundles carried away in Chapter 4, the description of the basic transformations and the general overall work with the moduli space is developed using the Mehta-Seshadri description of families of quasi-parabolic vector bundles. This is done because we want to explicitly untie the interactions of the underlying vector bundle, the parabolic structure and the weights. In particular, one of the key strategies used to engage the main problem was to consider the moduli spaces of parabolic bundles as moduli spaces of quasi-parabolic vector bundles which satisfied a stability condition and then consider different stability conditions for the same families of quasi-parabolic bundles. Thus, the choice of a description of the filtrations that was completely independent to the parabolic weights was needed. On the other hand, the analysis of the basic transformations (and, more precisely, the stability of parabolic bundles obtained from applying a basic transformation such as a Hecke or a dualization) was found to be more tractable when we approached it from other

points of view, such as the Simpson description. Overall, describing and analyzing the basic transformations from the point of view of all the previous formalisms (as well as other frameworks that will be described below) was a key point in the success of the stability of the resulting bundles.

Going back to the Simpson description, we can make a similar gluing procedure with the sheaves \widetilde{E}_α^x but, instead of extending all of them with \widetilde{E}_0^y , we can glue together the sheaves $E_{\alpha_x}^x$ for each choice of $\alpha_x \in \mathbb{R}$ and each $x \in D$, thus obtaining a filtration of i_*i^*E by subsheaves indexed by the multi-index $\alpha = (\alpha_{x_1}, \dots, \alpha_{x_n}) \in \mathbb{R}^{|D|}$. More precisely, we can also define parabolic vector bundles in the following way due to Mochizuki [Moc06] (See also [IS08])

Definition B.1.5 (Parabolic bundle Mochizuki). *A parabolic bundle on (X, D) is a collection of vector bundles E_α indexed by $\alpha = (\alpha_{x_1}, \dots, \alpha_{x_n}) \in \mathbb{R}^{|D|}$ together with inclusions of sheaves of \mathcal{O}_X -modules $E_\alpha \hookrightarrow E_\beta$ whenever $\alpha_x \geq \beta_x$ for all $x \in D$ ($\alpha \geq \beta$), such that*

1. *If $\delta^x = (\delta_y^x)$ is the multiindex with $\delta_x^x = 1$ and $\delta_y^x = 0$ for $y \neq x$, then for every multiindex α*

$$E_{\alpha+\delta^x} = E_\alpha(-x)$$

2. *For any α there exist $\varepsilon > 0$ such that for every multiindex δ with $0 \leq \delta < \varepsilon$ we have*

$$E_{\alpha-\delta} = E_\alpha$$

We will call $E = E_{(0, \dots, 0)}$ the underlying vector bundle of $\{E_\alpha\}$.

Observe that in this case

$$i_*i^*E = \bigcup_{\alpha \in \mathbb{R}^n} E_\alpha$$

Once again, taking restrictions or extensions to $U \cup \{x\}$ for each $x \in D$, it is clear that over a curve the filtration E_α indexed by the multi-index $\alpha = (\alpha_x)$ encloses the same information as the independent filtrations E_α^x given by Simpson's original Definition B.1.3. Nevertheless, in higher dimension there are subtle differences between how these approaches deal with the behavior of the filtrations at the intersection of two divisors. If $D = D_1 + \dots + D_n$ is now a divisor over a higher dimensional variety X and we want to treat each filtration over a extension $U \cup D_i$ independently, then we must impose a certain compatibility condition to how do the filtrations restrict to the intersection of the divisors (in the literature, these restricted filtrations are denoted as “compatible filtrations” or “locally Abelian parabolic bundles” [Moc06, IS08]).

Given sheaves $E_\alpha^{D_i}$ over X indexed by α for each $D_i \subset D$, we can define the multiindexed filtration

$$E_{\alpha_1, \dots, \alpha_n} = \bigcap_{i=1}^n E_{\alpha_i}^{D_i}$$

where the intersection is taken considering all sheaves $E_\alpha^{D_i}$ as subsheaves of i_*i^*E . Nevertheless, not every multiindex filtration can be obtained as a result of the previous process and the aforementioned local conditions on the intersections must be imposed in order to obtain a completely equivalent object [Moc06, Corollary 4.4] [IS08, Lemma 2.1].

Through the previous definitions a common theme is that the evolution of the filtration is more or less kept independently at each parabolic point. Even in the Mochizuki definition, if we fix all indexes of the multiindex and vary only one of the weighs we obtain a filtration whose quotients are supported on a component of the parabolic divisor. Contrary to this point of view, Maruyama and Yokogawa [MY92] define a parabolic structure as a filtration by subsheaves where the variation occurs globally on D , and is not localized at any particular component D_i .

Definition B.1.6 (Quasi-parabolic sheaf Maruyama-Yokogawa). *A quasi-parabolic sheaf on (X, D) is a torsion free sheaf E endowed with a decreasing filtration by subsheaves over X*

$$E = E_1 \supsetneq E_2 \supsetneq \dots \supsetneq E_{l+1} = E(-D)$$

A parabolic sheaf is a quasi-parabolic sheaf together with a set of real weights $0 \leq \alpha_1 < \alpha_2 < \dots < \alpha_l < 1$ called the parabolic weights. The polynomial $\chi(E_i/E_{i+1})(m)$ is called the multiplicity polynomial of α_i .

Observe that, since the quotients of successive steps of the filtration E_i/E_{i+1} are not supported on points anymore, instead of just giving a dimension it is necessary to give the full Hilbert polynomial of the corresponding torsion sheaf supported on D . In particular, if we work on a curve we must give a multi-rank identifying the dimension of the jump at each parabolic point $x \in D$.

Similarly as with the independent filtrations, we can alternatively codify both the filtration and the weights together using a left-continuous filtration indexed by a real parameter $\alpha \in \mathbb{R}$.

Definition B.1.7 (Parabolic sheaf Maruyama-Yokogawa). *A parabolic sheaf E_\bullet is a set of sheaves E_α on X indexed by $\alpha \in \mathbb{R}$ satisfying the following conditions*

- 1" For every $\alpha \geq \beta$, $E_\alpha \subseteq E_\beta$
- 2" For every $\alpha > 0$, there exists $\varepsilon > 0$ such that $E_{\alpha-\varepsilon} = E_\alpha$
- 3" For every α , $E_{\alpha+1} = E_\alpha(-D)$

We call $E = E_0$ the underlying sheaf of E_\bullet .

Once again, we have a filtration

$$i_* i^* E = \bigcup_{\alpha \in \mathbb{R}} E_\alpha$$

Clearly, we can interchange Definitions B.1.6 and B.1.7 simply taking

$$E_\alpha = E_i(-[\alpha]D)$$

where $1 \leq i \leq l$ is the only integer such that $\alpha_{i-1} < \alpha - [\alpha] \leq \alpha_i$, with the convention that $\alpha_0 = \alpha_l - 1$ and $\alpha_{l+1} = 1$. Moreover, if we have a filtered sheaf by a multiindex as the one given by Definition B.1.5, then we can associate it another one in the Maruyama-Yokogawa formalism taking

$$E_\alpha = E_{(\alpha, \alpha, \dots, \alpha)}$$

If X is a curve, this is enough to obtain an equivalence of categories. As the parabolic points are disjoint, the successive quotients of the filtration give us filtrations over $E|_x$ for each parabolic point (we just split the filtration obtained over $E|_D$ into each component). Nevertheless, as it happened with the comparison between the Simpson and the Mochizuki definitions, in this case it is not true that any filtered sheaf in the Maruyama-Yokogawa formalism could be recovered by a multiindex in dimension more than one. Once again, these definitions describe equivalent categories if we restrict ourselves to “locally abelian parabolic bundles” [IS08].

Finally, we shall briefly mention that this formalism was extended by Yokogawa [Yok95] to describe the more flexible category of \mathbb{R} -filtered \mathcal{O}_X -modules, which is the key to the completion of the category of parabolic vector bundles into an Abelian category.

B.2 Parabolic morphisms and exact sequences

Given parabolic vector bundles (E, E_\bullet) and (F, F_\bullet) with systems of weights α and β respectively, a morphism $\varphi : E \rightarrow F$ is called parabolic (respectively strongly parabolic) if it preserves the parabolic structure, i.e., if for every $x \in D$ and every $i = 1, \dots, l_{E,x}$ and $j = 1, \dots, l_{F,x}$ such that $\alpha_i(x) > \beta_j(x)$ (respectively $\alpha_i(x) \geq \beta_j(x)$)

$$\varphi(E_{x,i}) \subseteq F_{x,j+1}$$

We denote by $\text{PHom}((E, E_\bullet), (F, F_\bullet))$ the sheaf of local parabolic morphisms from (E, E_\bullet) to (F, F_\bullet) and write $\text{SPHom}((E, E_\bullet), (F, F_\bullet))$ for the subsheaf of strongly parabolic morphisms.

In particular, if (E, E_\bullet) is a parabolic vector bundle, an endomorphism $\varphi : E \rightarrow E$ is parabolic if for every $x \in D$ and every $i = 1, \dots, l_x$

$$\varphi(E_{x,i}) \subseteq E_{x,i}$$

We denote by $\text{PEnd}(E, E_\bullet)$ the sheaf of local parabolic endomorphisms of (E, E_\bullet) . Similarly, an endomorphism is strongly parabolic if for every $x \in D$ and every $i = 1, \dots, r$

$$\varphi(E_{x,i}) \subseteq E_{x,i+1}$$

We denote by $\text{SPEnd}(E, E_\bullet)$ the sheaf of strongly parabolic endomorphisms of (E, E_\bullet) . Clearly, the sheaves PHom and SPHom are subsheaves of the sheaf of morphisms Hom and they all coincide away from the parabolic points $D \subset X$. We will use a similar notation when we work on the formalism of \mathbb{R} -filtered sheaves of Yokogawa, if E_\bullet and F_\bullet are parabolic vector bundles, we will write $\text{PHom}(E_\bullet, F_\bullet)$ and $\text{SPHom}(E_\bullet, F_\bullet)$ to denote the sheaves of parabolic morphisms and strongly parabolic morphisms respectively.

In the formalism of \mathbb{R} -filtered sheaves (Definition B.1.7) the description of parabolic morphisms is particularly simple. A morphism of parabolic vector bundles is a morphism between the underlying vector bundles $f : E \rightarrow F$ such that for every α with $0 \leq \alpha \leq 1$

$$F(E_\alpha) \subseteq F_\alpha$$

We will denote by $f_\alpha = f|_{E_\alpha} : E_\alpha \rightarrow F_\alpha$ the induced morphism between the corresponding steps for the filtration for every $\alpha \in [0, 1]$. Observe that the periodicity conditions on E_α allows us, in fact, to define f_α for every $\alpha \in \mathbb{R}$.

Let E_\bullet , E'_\bullet and E''_\bullet be parabolic vector bundles. We say that they form a short exact sequence

$$0 \longrightarrow E'_\bullet \xrightarrow{f} E_\bullet \xrightarrow{g} E''_\bullet \longrightarrow 0$$

if and only if the underlying vector bundles form a short exact sequence

$$0 \longrightarrow E' \xrightarrow{f} E \xrightarrow{g} E'' \longrightarrow 0$$

and for each $\alpha \in [0, 1]$, the induced maps f_α and g_α form a short exact sequence

$$0 \longrightarrow E'_\alpha \xrightarrow{f_\alpha} E_\alpha \xrightarrow{g_\alpha} E''_\alpha \longrightarrow 0$$

B.3 Operations with parabolic vector bundles

Through this section we will use the Maruyama-Yokogawa formalism of \mathbb{R} -filtered sheaves for parabolic vector bundles (Definition B.1.7), as it simplifies enormously the definitions of the operations between parabolic vector bundles. Therefore, we will represent a parabolic vector bundle as a left continuous filtration $E_\bullet = \{E_\alpha\}_{\alpha \in \mathbb{R}}$. Let E_\bullet and F_\bullet two parabolic vector bundles. Let us denote by $E = E_0$ and $F = F_0$ the underlying vector bundles respectively, and let Λ_E and Λ_F denote the set of parabolic weights respectively, i.e., the set of $\alpha \in [0, 1)$ where the filtration E_\bullet (respectively F_\bullet) jumps. Similarly to the previous section, let $U = X \setminus D$ and let $i : U \hookrightarrow X$ be the inclusion.

B.3.1 Direct sum of parabolic vector bundles

The definition of direct sum is the expected one. As we have a filtration for each of the vector bundles, taking piecewise direct sums of the filtrations induce a filtration of the direct sum of the vector bundles. More precisely, let

$$G := i_* i^* E \oplus i_* i^* F = i_* i^* (E \oplus F)$$

Then the filtrations E_\bullet and F_\bullet induce an \mathbb{R} -filtration on G the following way, take

$$G_\alpha := E_\alpha \oplus F_\alpha$$

for all $\alpha \in \mathbb{R}$. It is clear that this filtration inherits the properties required for a parabolic structure. We call the parabolic vector bundle G_\bullet the direct sum of E_\bullet and F_\bullet and we will denote it as $E_\bullet \oplus F_\bullet$.

The underlying vector bundle is simply the sum of the underlying vector bundles, as

$$(E_\bullet \oplus F_\bullet)_0 = E_0 \oplus F_0 = E \oplus F$$

and the parabolic weights are the union of the parabolic weights of each parabolic vector bundle

$$\Lambda_{E_\bullet \oplus F_\bullet} = \Lambda_E \cup \Lambda_F$$

B.3.2 Parabolic tensor product

The tensor product of a parabolic vector bundle with a vector bundle is defined simply taking the tensor product of each element of the filtration. If V is a vector bundle, define

$$(E_\bullet \otimes V)_\alpha := E_\alpha \otimes V$$

It is clear that E_α is a left continuous \mathbb{R} -filtration that satisfies the conditions for a parabolic structure whose underlying vector bundle is $E \otimes V$ and whose parabolic weights are the same.

The definition of the tensor product between two parabolic vector bundles is a little bit more convoluted. Unlike the direct sum, we cannot simply take the direct product of each element of the filtration (“ $E_\alpha \otimes F_\alpha$ ”). For example, this would break the periodicity condition on the filtration, as $E_1 \otimes F_1 = E(-D) \otimes F(-D) = (E \otimes F)(-2D) \neq (E \otimes F)(-D)$. Moreover, if we took F_\bullet to be a trivial filtration on F (i.e., $\Lambda_F = \{0\}$), we would like to recover the natural definition of the product of a parabolic vector bundle with a vector bundle. This gives us a clue which indicates that a natural definition of a product of filtered vector bundles should take all steps of both the filtrations into account, as well as the values of the weights.

Similarly to the direct sum, let

$$G := i_* i^* E \otimes i_* i^* F = i_* i^* (E \otimes F)$$

For each $\gamma \in \mathbb{R}$, let G_γ be the subsheaf of G generated by all the subsheaves $E_\alpha \otimes F_\beta$, with $\alpha + \beta \geq \gamma$. Observe that for each α and β we have an isomorphism

$$j^{\alpha, \beta} : E_{\alpha+1} \otimes F_{\beta-1} = E_\alpha \otimes \mathcal{O}_X(-D) \otimes F_\beta \otimes \mathcal{O}_X(D) \cong E_\alpha \otimes F_\beta$$

and, moreover, we know that the filtrations E_α and F_β are non-increasing and semi-continuous so the sheaf G_γ is always generated by a finite number of combinations of tensor products $E_\alpha \otimes F_\beta$

$$G_\gamma = \sum_{\alpha+\beta \geq \gamma} E_\alpha \otimes F_\beta = \sum_{0 \leq \alpha < 1} E_\alpha \otimes F_{\gamma-\alpha} = \sum_{0 \leq \beta < 1} E_{\gamma-\beta} \otimes F_\beta$$

With this description, it is clear that the set of weights for G_\bullet is formed by the sums of weights in Λ_E and Λ_F , shifted to $[0, 1)$

$$\Lambda_G = \{\alpha + \beta | \alpha \in \Lambda_E, \beta \in \Lambda_F, \alpha + \beta < 1\} \cup \{\alpha + \beta - 1 | \alpha \in \Lambda_E, \beta \in \Lambda_F, \alpha + \beta \geq 1\}$$

Yokogawa [Yok95] gives an alternative presentation of this filtration. Let $i_E^{\alpha_1, \alpha_2} : E_{\alpha_1} \rightarrow E_{\alpha_2}$ be the inclusion of subsheaves. Then for each $\gamma \in \mathbb{R}$ define the sheaf

$$G_\gamma = \left(\bigoplus_{\alpha+\beta=\gamma} E_\alpha \otimes F_\beta \right) / R_\gamma$$

where $R_\gamma \subset \bigoplus_{\alpha+\beta=\gamma} E_\alpha \otimes F_\beta$ is the \mathcal{O}_X -submodule generated locally by the following types of sections

1. For each $x \in E_{\alpha_1}$ and $y \in F_{\beta_2}$ with $\alpha_1 + \beta_1 = \alpha_2 + \beta_2 = \gamma$

$$i_E^{\alpha_1, \alpha_2}(x) \otimes y - x \otimes i_F^{\beta_1, \beta_2}(y)$$

2. For each $x \in E_{\alpha} \otimes F_{\beta}$ with $\alpha + \beta = \gamma$

$$x - j^{\alpha, \beta}(x)$$

Then if $\gamma \geq \gamma'$ we have a canonical inclusion

$$i_G^{\gamma, \gamma'} : G_{\gamma} \rightarrow G_{\gamma'}$$

defined as follows. Let $x \in E_{\alpha}$ and $y \in F_{\beta}$ with $\alpha + \beta = \gamma$. Then

$$i_G^{\gamma, \gamma'}(x \otimes y \mod R_{\gamma}) = i_E^{\alpha, \gamma' - \alpha}(x) \otimes y \mod R_{\gamma'} = x \otimes i_F^{\beta, \gamma' - \beta}(y) \mod R_{\gamma'}$$

Observe that the last equality is a consequence of (1), so the map $i_G^{\gamma, \gamma'}$ is a well defined inclusion. Moreover, it is straightforward to check that

$$\bigoplus_{\alpha + \beta = \gamma + 1} E_{\alpha} \otimes F_{\beta} = \left(\bigoplus_{\alpha + \beta = \gamma} E_{\alpha} \otimes F_{\beta} \right) \otimes \mathcal{O}_X(-D)$$

and

$$R_{\gamma+1} = R_{\gamma} \otimes \mathcal{O}_X(-D)$$

so $G_{\gamma+1} = G_{\gamma} \otimes \mathcal{O}_X(-D)$ and G_{\bullet} is a parabolic vector bundle.

A direct computation shows that both definitions for the sheaf G_{γ} coincide when we consider the sheaves $E_{\alpha} \otimes F_{\beta}$ as subsheaves of $i_* i^* E \otimes i_* i^* F = i_* i^*(E \otimes F)$. We will call G_{\bullet} the parabolic tensor product of E_{\bullet} and F_{\bullet} and we will denote it by $E_{\bullet} \otimes F_{\bullet}$. If we are working with other formalisms for parabolic vector bundles we will use a similar notation and denote by $(E, E_{\bullet}) \otimes (F, F_{\bullet})$ the tensor product of the parabolic vector bundles (E, E_{\bullet}) and (F, F_{\bullet}) .

Observe that, with this definition, if F_{\bullet} is the trivial structure on a vector bundle F , then

$$(E_{\bullet} \otimes F_{\bullet})_{\gamma} = E_{\gamma} \otimes F_0 = E_{\gamma} \otimes F$$

so we recover the natural description of the tensor product of a parabolic vector bundle with a vector bundle.

On the other hand, notice that the underlying vector bundle of $E_{\bullet} \otimes F_{\bullet}$ may not coincide with $E \otimes F$, as

$$(E_{\bullet} \otimes F_{\bullet})_0 = \sum_{0 \leq \alpha < 1} E_{\alpha} \otimes F_{-\alpha}$$

This sum always includes $E_0 \otimes F_0$, so the sheaf $E_{\bullet} \otimes F_{\bullet}$ contains $E \otimes F$ as a subsheaf. Nevertheless, if there exist weights $\alpha \in \Lambda_E$ and $\beta \in \Lambda_F$ with $\alpha + \beta \geq 1$, then

$$E_{\alpha-1} \otimes F_{\beta} = E_{\alpha} \otimes F_{\beta-1} \subset (E_{\bullet} \otimes F_{\bullet})_0$$

We know that $E \subsetneq E_{\alpha-1}$ if $\alpha > 0$ and otherwise $\beta > 0$, so $F \subsetneq F_{\beta-1}$. Anyway, $E_{\alpha-1} \otimes F_{\beta} \not\subset E \otimes F$, so the underlying vector bundle of $E_{\bullet} \otimes F_{\bullet}$ does not coincide with $E \otimes F$.

Let us end this section with a couple of properties of the tensor product of parabolic sheaves

Proposition B.3.1 (c.f. [Bis03, Yok95]). *The tensor product is associative and distributive with respect to direct sum, i.e., for every parabolic vector bundles E_\bullet , F_\bullet and G_\bullet we have*

$$E_\bullet \otimes (F_\bullet \otimes G_\bullet) = (E_\bullet \otimes F_\bullet) \otimes G_\bullet$$

and

$$E_\bullet \otimes (F_\bullet \oplus G_\bullet) = (E_\bullet \otimes F_\bullet) \oplus (E_\bullet \otimes G_\bullet)$$

as a parabolic vector bundle.

Proposition B.3.2 ([Yok95, Propositions 3.2 and 3.3]). *The parabolic tensor product is right-exact, i.e., if E_\bullet , E'_\bullet , E''_\bullet and F_\bullet are parabolic sheaves such that*

$$0 \longrightarrow E'_\bullet \longrightarrow E_\bullet \longrightarrow E''_\bullet \longrightarrow 0$$

is an exact sequence, then the sequence

$$E'_\bullet \otimes F_\bullet \longrightarrow E_\bullet \otimes F_\bullet \longrightarrow E''_\bullet \otimes F_\bullet \longrightarrow 0$$

is exact. Moreover, if F_\bullet is locally free then the map $E'_\bullet \otimes F \rightarrow E_\bullet \otimes F$ is injective. The functor $- \otimes F_\bullet$ is exact if and only if F_\bullet is locally free.

B.3.3 Shift of a parabolic vector bundle

Let E_\bullet be a parabolic sheaf on (X, D) and let $\varepsilon \in \mathbb{R}$ be a real number. Then we can construct another \mathbb{R} -filtered sheaf simply translating or “shifting” the filtration by the constant number ε . We define the parabolic sheaf $E_\bullet[\varepsilon]$ as the filtration such that for every $\alpha \in \mathbb{R}$

$$(E_\bullet[\varepsilon])_\alpha = E_{\alpha+\varepsilon}$$

By construction the parabolic weights of the new sheaf are obtained by subtracting the constant ε and then shifting back to $[0, 1)$, i.e.,

$$\Lambda_{E[\varepsilon]} = \{\alpha + \varepsilon - \lfloor \alpha + \varepsilon \rfloor \mid \alpha \in \Lambda_E\}$$

Observe that the underlying vector bundle of $E_\bullet[\varepsilon]$ is not necessarily E . In fact, if Λ_E contains a parabolic weight α such that $\alpha + \varepsilon \notin [0, 1)$, then the underlying vector bundle is not E , but a combination of a Hecke transformation of E over D and tensorization by an appropriate line bundle of the form $\mathcal{O}_X(F)$ with F supported on D . See Section 4.4 for a full description of the relation between shifting and Hecke transformations.

Observe that if $\varepsilon > 0$, then $E_\bullet[\varepsilon] \subset E_\bullet$, as for every $\alpha \in \mathbb{R}$ we have an inclusion

$$E_\alpha[\varepsilon] = E_{\alpha+\varepsilon} \subset E_\alpha$$

Moreover, if $\varepsilon \in \mathbb{Z}$, then

$$E_\bullet[\varepsilon] = E_\bullet(-\varepsilon D)$$

Thus, the shifting operation can be used to induce a parabolic structure on the sheaf of parabolic morphisms between two parabolic sheaves $\text{PHom}(E_\bullet, F_\bullet)$. We

simply have to notice that $\mathrm{PHom}(E_\bullet, F_\bullet[\varepsilon])$ gives us a filtration of $\mathrm{PHom}(E_\bullet, F_\bullet)$ as ε ranges in the interval $[0, 1)$. Take

$$\mathrm{PHom}(E_\bullet, F_\bullet)_\alpha := \mathrm{PHom}(E_\bullet, F_\bullet[\alpha])$$

If $\alpha \geq \beta$ and we take $\varepsilon = \beta - \alpha$, then we have an inclusion

$$F_\bullet[\alpha] = F_\bullet[\beta][\varepsilon] \subset F_\bullet[\beta]$$

and by composition, it induces an inclusion

$$\mathrm{PHom}(E_\bullet, F_\bullet)_\alpha = \mathrm{PHom}(E_\bullet, F_\bullet[\alpha]) \hookrightarrow \mathrm{PHom}(E_\bullet, F_\bullet[\beta]) = \mathrm{PHom}(E_\bullet, F_\bullet)_\beta$$

Therefore, PHom gives us an internal Hom functor in the category of parabolic sheaves. Moreover, we can state the following adjacency property

Proposition B.3.3 ([Yok95, Proposition 3.5]). *The parabolic tensor product and the parabolic Hom are adjoint, i.e., for every E_\bullet , F_\bullet and G_\bullet there is a natural isomorphism of parabolic sheaves*

$$\mathrm{PHom}(E_\bullet \otimes F_\bullet, G_\bullet)_\bullet \cong \mathrm{PHom}(E_\bullet, \mathrm{PHom}(F_\bullet, G_\bullet)_\bullet)_\bullet$$

Finally, I would like to remark that the shifting operation can be rewritten in terms of a parabolic tensor product. This correspondence was developed in more generality for the Simpson's formalism in Section 4.4. Assume without loss of generality that $\varepsilon \in [0, 1)$ (otherwise, take out the integer part and tensorize with the appropriate power of $\mathcal{O}_X(-D)$). Let $(\mathcal{O}_X, \mathcal{O}_X^\varepsilon) = \mathcal{O}_{X, \bullet}^\varepsilon$ be the trivial vector bundle endowed with the trivial parabolic structure on D whose only weight is ε . Then for every parabolic bundle E_\bullet

$$E_\bullet[-\varepsilon] \cong E_\bullet \otimes \mathcal{O}_{X, \bullet}^\varepsilon$$

because for every α

$$E_\alpha[-\varepsilon] = E_{\alpha+\varepsilon} = E_{\alpha-\varepsilon} \otimes \mathcal{O}_{X, \varepsilon}^\varepsilon = (E_\bullet \otimes \mathcal{O}_{X, \bullet}^\varepsilon)_\alpha$$

B.3.4 Parabolic dual

We will summarize the definition of dual of an \mathbb{R} -filtered sheaf by Yokogawa [Yok95]. Some additional comments and details on the construction can be found in [Bis03]. Let E_\bullet be a parabolic sheaf. For every $\alpha \in \mathbb{R}$, let us denote by $E_{\alpha+}$ the inductive limit

$$E_{\alpha+} = \lim_{\varepsilon \rightarrow 0^+} E_{\alpha+\varepsilon}$$

Clearly $E_{\alpha+}$ is a filtration of $i_* i^* E$ by subsheaves and satisfies the usual periodicity condition

$$E_{(\alpha+1)+} = E_{\alpha+}(-D)$$

for each $\alpha \in \mathbb{R}$. On the other hand, as E_α is left-continuous, then $E_{\alpha+}$ is right-continuous. With this in mind, we define the parabolic dual E_\bullet^* of E_\bullet as the \mathbb{R} -filtered vector bundle given by

$$E_\alpha^* := E_{(-1-\alpha)+}^\vee$$

Observe that if we restrict to U , then for every α we have a natural isomorphism $E_\alpha^*|_U = E|_U^\vee$, so for every $\alpha \in \mathbb{R}$, E_α^* lays naturally as a subsheaf

$$E_\alpha^* \subset i_* i^* E^\vee$$

If $\alpha \geq \beta$ then $-1 - \alpha \leq -1 - \beta$, so we have an inclusion

$$E_{(-1-\beta)^+} \hookrightarrow E_{(-1-\alpha)^+}$$

Taking the dual, we get a map $E_\alpha^* \rightarrow E_\beta^*$ which, regarding both sheaves inside $i_* i^* E^\vee$, gives us an inclusion $E_\alpha^* \subset E_\beta^*$. Therefore, E_\bullet^* is a parabolic sheaf over (X, D) and we will call it the parabolic dual (or simply the dual) of E_\bullet . When we work with other parabolic vector bundle formalisms (such as the Mehta-Seshadri or Simpson's definitions), we will analogously denote the parabolic dual of the parabolic vector bundle (E, E_\bullet) by $(E, E_\bullet)^*$.

Observe that the underlying parabolic vector bundle of E_\bullet^* is not E^\vee . In fact, the underlying vector bundle depends on the parabolic weights. If $0 \notin \Lambda_E$, then by construction

$$E_0^* = E_{-1}^\vee = (E_0(D))^\vee = E^\vee(-D)$$

so the underlying vector bundle of E_\bullet^* is simply $E^\vee(-D)$ if $0 \notin \Lambda_E$. Otherwise, if $0 \in \Lambda_E$, then by semicontinuity the underlying bundle of E_\bullet^* coincides with the subsheaf

$$E_0^* = E_{-1+\alpha_1}^\vee = E_1^\vee(-D)$$

so it can be found as a Hecke transformation of $E^\vee(-D)$. By construction it is clear that every positive parabolic weight $\alpha \in \Lambda_E \setminus \{0\}$ representing a jump in the filtration of E_\bullet gives rise to a jump in the filtration of E_\bullet^* at $1 - \alpha$. Moreover, if 0 was an original jump, then after taking the limit it remains as a jump. Therefore

$$\Lambda_{E_\bullet^*} = (\Lambda_E \setminus \{0\}) \cup (\Lambda_E \cap \{0\})$$

In any case, from the definition it is straightforward to check that for every locally free parabolic vector bundle

$$(E_\bullet^*)^* = E_\bullet$$

Moreover, the following Propositions show that this notion of dual works particularly with the previously described tensor product, as it allows us to recover the internal Homs in the category of parabolic sheaves.

Proposition B.3.4 ([Yok95, Lemma 3.6]). *If E_\bullet is locally free and F_\bullet is a parabolic sheaf then there are canonical isomorphisms*

$$E_\bullet^* \otimes F_\bullet \cong \mathrm{PHom}(E_\bullet, F_\bullet)_\bullet$$

In particular, we find that the natural definition of the dual via the internal Hom agrees with the one described at the level of filtrations, in the sense that for every parabolic vector bundle E_\bullet we have

$$E_\bullet^* = E_\bullet^* \otimes \mathcal{O}_X \cong \mathrm{PHom}(E_\bullet, \mathcal{O}_X)_\bullet$$

Moreover, the expected compatibility between tensor product and dual holds.

Corollary B.3.5. *If E_\bullet and F_\bullet are parabolic vector bundles then*

$$(E_\bullet \otimes F_\bullet)^* \cong E_\bullet^* \otimes F_\bullet^*$$

Proof. Combining Proposition B.3.3 and Proposition B.3.4 we have

$$(E_\bullet \otimes F_\bullet)^* \cong \mathrm{PHom}(E_\bullet \otimes F_\bullet, \mathcal{O}_X) \cong \mathrm{PHom}(E_\bullet, \mathrm{PHom}(F_\bullet, \mathcal{O}_X)) \cong \mathrm{PHom}(E_\bullet, F_\bullet^*) \cong E_\bullet^* \otimes F_\bullet^*$$

□

B.4 Parabolic version of Serre duality

Similarly to the non-parabolic case, the category of parabolic vector bundles over a smooth variety admits a version of Serre duality. For simplicity (and because it is the only case needed in the thesis), we present here a simplified version for curves. The proof of the complete result for smooth varieties of arbitrary dimension can be found in [Yok95].

Proposition B.4.1 ([Yok95, Proposition 3.7]). *Let E_\bullet and F_\bullet be parabolic vector bundles over a non-singular curve X . Then there are natural isomorphisms*

$$H^0(X, \mathrm{PHom}(E_\bullet, F_\bullet \otimes K_X(D))) \cong H^1(X, \mathrm{SPHom}(F_\bullet, E_\bullet))^\vee$$

$$H^1(X, \mathrm{PHom}(E_\bullet, F_\bullet \otimes K_X(D))) \cong H^0(X, \mathrm{SPHom}(F_\bullet, E_\bullet))^\vee$$

Moreover, if we replace F_\bullet by $F_\bullet \otimes K_X^{-1}(-D)$ and dualize, we obtain the analogous relations

$$H^0(X, \mathrm{PHom}(E_\bullet, F_\bullet)) \cong H^1(X, \mathrm{SPHom}(F_\bullet, E_\bullet) \otimes K_X(D))^\vee$$

$$H^1(X, \mathrm{PHom}(E_\bullet, F_\bullet)) \cong H^0(X, \mathrm{SPHom}(F_\bullet, E_\bullet) \otimes K_X(D))^\vee$$

Finally, if we take the trivial parabolic structure on the bundles we recover the usual Serre duality.

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